How Substitutable are Fixed Factors in Production?
Evidence from Pre-Industrial England

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Abstract

Whether fixed factors such as land constrain per capita income growth depends crucially on two variables: the substitutability of fixed factors in production, and the extent to which innovation is biased towards land-saving technologies. This paper attempts to quantify both. Using the timing of plague epidemics as an instrument for labor supply, I estimate the elasticity of substitution between fixed and non-fixed factors in pre-industrial England to be significantly less than one. I also find evidence that denser populations — and hence higher land scarcity — induced innovation towards land-saving technologies. Finally, by varying this elasticity of substitution in a number of well-known models of the Malthusian economy, I show that the elasticity of substitution predicts the speed by which the demographic transition occurs, the order in which countries will transition, and the divergence of income levels before the transition. In addition, I show this elasticity has important ramifications for the question of how population size affects economic development.

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1 Introduction

One of the least understood aspects of economic development is the extent to which fixed factors constrain economic growth. The debate has a long history. Malthus (1798) predicted that humanity, if left unchecked, would breed itself into poverty due to its inability to produce food proportional to a growing population in the face of a fixed amount of arable land. More recently, many have predicted the world will run out of oil, coal, clean water, timber, arable land, or other essential natural resources, with disastrous economic consequences. While some scholars maintain that the world is heading towards poverty due to the fixedness of natural resources, others such as and argue natural resources pose little if any restrictions on boundless economic growth (see Boserup 1965, Ehrlich 1968, Simon 1981, and Brown 2004).

The extent to which fixed factors constrain growth is not only important to our future, but also to understanding our past. Many economists, including Lee (1973), Galor (2005) and Clark (2007a) argue that the history of world population before 1750 is well summarized by Malthusian population dynamics. The role of fixed factors in production is a crucial element of this theory. For example, the quantity of fixed factors and their substitutability with other factors of production affects population size and dynamics around the Malthusian steady state. These dynamics are key in understanding the large and active body of literature exploring the escape from the Malthusian steady state and the takeoff to modern growth in Europe during the 19th century (see Lucas 2004, Galor and Weil 1999 and 2000, Hansen and Prescott 2002, Doepke 2004, Fernandez-Vilaverde 2011, and Voigtländer and Voth 2013a).

Whether fixed factors limit growth depends crucially on two variables. The first is the nature of productivity growth, and the second is the elasticity of substitution between fixed and non-fixed factors.

Productivity growth can help alleviate resource constraints in two ways. First, any type of productivity growth can help overcome the constraining effects of a fixed factor. For example, sustained income growth is possible in the presence of a fixed factor, given a high enough rate of productivity growth. Predicting the future rate of productivity growth, however, is an extremely difficult, if not impossible, task which I do not explore here in this paper.

Secondly, the direction of innovation is important in overcoming resource constraints. If productivity growth is not Hicks-neutral, then whether future innovations are fixed or non-fixed factor saving is important. There are many reasons to think that future productivity growth may take place in the fixed factor sector. Hicks (1932) introduced the concept of “induced innovation”, or the tendency for new innovations to save on scarce resources. If technological progress will be fixed-factor saving, then fixed factors may not be a large barrier to growth.
A high elasticity of substitution between fixed and non-fixed factors also governs the speed by which the marginal product of non-fixed factors fall as these factors accumulate. This can lead to higher levels of income per capita given the same fixed factor endowment. The value of this elasticity is especially important in two types of models within economics. First, models of the effect of population size on income per capita can have very different results depending on this elasticity. For example, models which estimate the gain in income from a productivity-enhancing health intervention which also saves lives can also show a decrease in income if this elasticity is too low. Or models which predict a gain in income for a family planning program can have those gains erased if the elasticity is too high.

The second group of models for which this elasticity of substitution plays an important role are models of the demographic transition and takeoff from Malthusian stagnation. The Cobb-Douglas production function, a highly popular description of production technology among economists, implicitly assumes this elasticity to be unity. Virtually every major study explaining the Malthusian regime or the demographic transition has used an elasticity of substitution of one (see Galor and Weil 2000, Stokey 2001, Hansen and Prescott 2002, Doepke 2004, Lucas 2004, Strulik and Weisdorf 2008, and Voigtländer and Voth 2013a). The elasticity of substitution can affect a range of outcomes in these models, such as the speed of the demographic transition, the level of pre-industrial income, the speed by which factors shift from the agricultural to the industrial sector, how intensely wages react to population shocks, how quickly wages fall back to the Malthusian steady state after a productivity shock, steady state levels of population density, and predictions of which countries should make the demographic transition first.

However, despite the importance of this parameter, there are few – if any – reliable estimates of this elasticity. A few economists have attempted simple analyses within larger papers on other topics where this elasticity mattered, but no serious attempt has been made at its estimation. A common method used to estimate this elasticity is to regress shares of income paid to factors of production (or alternatively the rental rate paid to factors of production) on factor quantities. However, since these factor quantities are endogenous to their rental rates, these estimates are not well identified. In addition, data on factor shares are notoriously difficult to quantify, leading to imprecise estimates.

The first attempt was by Nordhaus and Tobin (1972), who used data on shares of income paid to land and labor from the U.S. in the first half of the 20th century and find that the elasticity of substitution is about 2. However, they only use 10 observations, and therefore they cannot estimate the elasticity with any certainty. The most cited estimate is from Hayami and Ruttan (1985), who look at a cross section of the agricultural sector in 30

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1 Nordhaus and Tobin fail to reject the elasticity is $\infty$, but also fail to reject 1.
developed countries and test whether the elasticity is unity and fail to reject. However, they do not measure the elasticity of fixed and non-fixed factors, but rather the elasticity of labor and all other factors, then make the assumption that land is the only factor. This is a difficult assumption to make, since capital is heavily used in agriculture in the developed world. Their study also suffers from a small sample size and poor identification.

Given the paucity of good estimates for the elasticity, recently Ashraf et al (2008) use data on the natural resource share in national income from Caselli and Feyrer (2007) for estimation. They find a value of 2.35 and can reject unity. While an improvement over previous studies in regards to sample size, Ashraf et al rely heavily on imputed data. First, they use imputed data on land’s share of income from Caselli and Feyrer (2007). They also impute the quantity of the fixed factor rather than measuring it directly, due to the fact that there is no obvious unit of measure for “fixed factors”. Finally, as in previous studies, the authors use a simple OLS regression with no attempt to overcome the endogeneity issues mentioned previously.

Weil and Wilde (2009) improve upon the work of Ashraf et al. by using a set of indicators on natural resource stock per capita, and use instruments for income. Depending on the model specification, they find an elasticity which varies from 1.5 to 5, with most of the estimates being around two.

All of these studies measure how the share of national income paid to fixed-factors (or alternatively the rental rate) changes as non-fixed factors accumulate to estimate this elasticity. However, none of the studies accounts for the fact that in an open economy, factor price ratios across countries tend to equalize to the world rate. Therefore, the estimation will be biased upward, since the observed change in factor rent ratios should be zero (or at least smaller than they would be in autarky) as non-fixed factors accumulate in an open economy. This implies that the estimation of the elasticity of substitution between fixed and non-fixed factors is better achieved in closed economies.

In addition, the most useful estimates of the elasticity of substitution will come from economies in which fixed factors play an important role – namely developing countries. If we truly want to use an estimate of the elasticity which is useful in explaining the transition from an agricultural economy to an industrial one, or to explain how differing elasticities of substitution may change growth dynamics or the effect of population size on income per capita in the developing world, it would be nice to estimate this elasticity for a set of agrarian economies.

In this paper, I estimate this elasticity of substitution in a closed, agrarian economy – pre-industrial England from 1200-1750. I use changes in labor supply via population size using data on factor rents from Clark (2007b) in a simple CES framework. As in the previous
labor ratio to uncover the elasticity of substitution. However, I overcome the issue of endogeneity by instrumenting for the land-labor ratio with the timing of plagues in England. As I discuss in section 2.2 below, while the size of the mortality caused by a plague incident is affected by the land-labor ratio, the timing of plagues is likely an exogenous source of variation in population levels, and can be traced back to specific exogenous events such as the arrival of a ship filled with plague-laden rats from an external port.

In addition to my estimation of the elasticity of substitution between fixed and non-fixed factors, my methodology also allows me to estimate the degree of factor-induced productivity growth. If technological progress is induced, then the land-labor ratio should not only affect the level of the ratio of factor rents via the elasticity of substitution, but also their growth rate via a changing ratio of factor-specific productivities. Using a model which incorporates both of these effects, I can control for and quantify both of these phenomena. This study is the first to my knowledge which can put a number on the extent to which induced innovation occurred in pre-industrial England.

Finally, I look at how my results for the elasticity of substitution and induced innovation impact well known economic models. These models fall into two categories. First, I show how changes in the elasticity of substitution would change dynamics in models of the transition from the Malthusian steady state to modern growth such as Galor and Weil (1999, 2000), Hansen and Prescott (2002), and Voigtländer and Voth (2013a). Second, I comment on how the elasticity of substitution would affect models of the effect of population size on development levels generally.

I find that the elasticity of substitution between land and other factors over this period in England was about 0.6. This implies that the elasticity of substitution is much smaller than previously thought, since most models addressing the issue of population and the demographic transition implicitly assume an elasticity of 1. This is a novel finding, because the share of income paid to land has been falling since at least the early 1900s (at least in the developed world), which is generally a result of an elasticity of substitution greater than one. This also implies that any short-run deviations of population size and income from their Malthusian steady state were shorter and smaller than previously modeled. In addition, since I estimate the elasticity of substitution over a long period (550 years), this implies that the elasticity I am measuring is a long-run elasticity. In the short run, this elasticity must have been even smaller, implying that land was even more constraining of a factor over shorter periods of time, such as an individual’s lifetime.

I also find evidence for factor-biased and induced technological change. Specifically, I find that the difference between the annual growth rates of land- and labor-augmenting
productivity was 0.1% higher per million additional persons – implying higher populations and therefore higher population densities induced innovation towards land-saving technologies. This implies that a doubling of population density in England from its year 1500 level raises the difference in the growth rates of land- and labor-enhancing productivity by 0.22% per year. Many economic historians have noted that technological changes over this period appeared to be induced, but this is the first study to empirically test and quantify this phenomenon.

I also find that smaller elasticities of substitution slow down the escape from Malthusian stagnation in models which depend on achieving a certain level of population to transition (such as Galor and Weil 1999 and 2000) and in models which rely on shifting production away from the agricultural sector due to biased technological progress (such as Hansen and Prescott 2002). However, I find that small elasticities of substitution should lead to faster transitions in models which rely on achieving a certain level of income to escape from the Malthusian steady state (such as Voigtländer and Voth 2013a and Stokey 2001). Since the timing of the transition to modern growth is highly correlated to development today, differences in the elasticity of substitution between land and labor across countries in the pre-industrial era (perhaps because of different geological suitability for certain crops in some areas of the world) could partially explain the great divergence in income we see across countries today.

Finally, I find that smaller elasticities of substitution should lead to a larger negative effect of population growth on income in the developing world. Interventions in developing countries which affect population size can have widely varying results based on the elasticity of substitution. For example, a health intervention which both saves lives and increases worker productivity could have a positive or negative effect on income depending on the elasticity of substitution. In addition, this large elasticities of substitution decrease the benefits of interventions to reduce fertility, such as family planning programs.

The paper continues as follows: Section 2 outlines the basic model and results. Section 3 augments the model to account for biased technological progress and provides results. Section 4 looks at the implications of different elasticities of substitution in economic models. Section 5 concludes.

2 Model with Hicks-Neutral Technology

Consider the following constant elasticity of substitution production function:

\[ Y_t = A_t \left[ (\psi X_t)^{\rho} + ((1 - \psi)N_t)^{\rho} \right]^{\frac{1}{\rho}}, \]  

(1)
where $X_t$ is a fixed factor, $N_t$ is a non-fixed factor, $Y_t$ is output, and $A_t$ is total factor productivity, $\psi \in (0,1)$ is a scale parameter, and $t$ indexes time. Notice that technology in this case is Hicks-neutral. Later in the paper I will relax the assumption of Hicks-neutral technology, which will allow me to estimate the magnitude of induced innovation.

The elasticity of substitution between $X_t$ and $N_t$ is

$$\sigma = \frac{1}{1 - \rho}.$$  

Taking the ratio of the marginal products of $X_t$ and $N_t$ (denoted $r_t^X$ and $r_t^N$), and taking logs yields

$$\ln \left( \frac{r_t^X}{r_t^N} \right) = -\frac{1}{\sigma} \ln \left( \frac{\psi}{1 - \psi} \right) - \frac{1}{\sigma} \ln \left( \frac{X_t}{N_t} \right). \tag{2}$$

In principle I can estimate the elasticity of substitution between fixed and non-fixed factors by regressing the log ratio of fixed factor rent and wages on a constant and the land-labor ratio.

There are several difficulties in estimating this equation directly. First, the quantities of $X$ and $N$ are potentially endogenous to their rental rates, and therefore estimation via OLS can yield biased results. Second, data on the quantity of factors in the economy are difficult to measure, especially in the historical context, and therefore may lead to attenuation bias in estimation. Third, there is no obvious unit of measure to combine different types of natural resources to create a value for the fixed factor stock $X_t$.\footnote{The most logical solution – to convert the value of the stock of each resource into dollar terms to obtain the total value of $X$ – is incorrect since resource values are generally obtained by merely capitalizing the stream of resource rents. Therefore, the dependent and independent variables would measure the same thing, and the estimates would be meaningless.}

Measurement issues are especially acute in the context of pre-industrial England, since there is little, if any, data on land use, capital stocks and rents, human capital attainment, or even labor force back to 1200. Therefore, I make two assumptions to simplify the baseline analysis:

1. The total endowment of natural resources, $X_t$, is constant over the entire sample.

2. The only non-fixed factor in the economy is labor, and labor is a fixed fraction of the population. Put mathematically, $N_t = L_t = \psi P_t$, where $P_t$ and $L_t$ are population and labor supply respectively at time $t$.

The first assumption side-steps the problem of measuring fixed resources. This allows $X_t = \bar{X}$ to become part of the constant term in my regression equation. The second assumption allows
me to use data on factors I can easily acquire for the period 1200-1750 – namely population, wages, and land rents – while forgetting about physical and human capital and their returns. Later in this section I will discuss each assumption and analyze what happens when each is relaxed.

Given these assumptions, we can rewrite (6) as:

$$\ln \left( \frac{r_t}{w_t} \right) = \alpha + \frac{1}{\sigma} \ln (P_t),$$

where $\alpha = -\frac{1}{\sigma} \ln \left( \frac{\psi_1}{1-\psi} \right) - \frac{1}{\sigma} \ln (\bar{X})$, $w_t$ is the wage, and $r_t$ is land rent. This will form our basic regression equation, which will be augmented with more factors of production and modified to allow for changes in factor-specific technological progress later in the paper.

### 2.1 Basic Data and OLS Results

The data on factor rents (wages and land rent) come from Clark (2001, 2002, 2007b).\(^3\) The unit of observation is the decade. Clark’s land rental income series is based on several data sets. For the period 1200-1500, he mainly uses manorial accounts and manor court records which record income from leases for a landholder (see Clark (2001) for more details). This is beneficial since his series is mostly based on actual transactions of land rentals rather than imputed from land prices, although some land price data is used for the period 1200-1320. For 1500-1870, he uses rental values of lands from charitable trusts. Although each of these samples are geographically diverse, they are not nationally representative since more densely populated areas will have disproportionately more data. Therefore, Clark calculates a national rent index by applying weights for regions, plot size, amount of common land, type of land (farmland, meadow, etc.) and population densities (see Clark 2002 for more details).\(^4\)

Nominal wages are determined by similar records on payments to hired farmhands and builders, which are converted to real terms by adjusting by the cost of a bundle of agricultural goods. The data and methods Clark uses to obtain this series are too detailed to be outlined in full here, but can be found in Clark (2007b). The data for population are also from Clark, who in turn used Wrigley et al (1997) for 1540-1870, and Hatcher (1977), Poos (1985), and Hallam (1972) for 1200-1540.

Figure 1 plots the wage and land rent series over time, in addition to the capital rent series.

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\(^3\)Since land was the main fixed factor in pre-industrial England, I use the rental rate on land as a proxy for the rental rate of fixed factors.

\(^4\)Clark graciously provided me with a revised version of his rental rate data from 1200-1800. Therefore, data I use may differ slightly from his published version.
from Clark which will be introduced and used later. Figure 2 plots the ratio of land rents to wages from Figure 1 to form a time series of our dependent variable. This is contrasted with population size, our independent variable of interest. It is immediately clear that the two series follow each other quite closely for the period 1200-1750. This is especially evident around 1348, when the Black Death sharply lowered population – the ratio of rental rates immediately followed. In the late 18th century, however, the relationship breaks down as population and factor prices begin to explode with the beginning of the industrial revolution.

Interestingly, it has been noted that the Malthusian model of the economy broke down almost as soon as Malthus wrote it. Not only did wages explode despite population growth, but factor prices began to be driven by forces other than population density, such as trade. O’Rourke and Williamson (2005) note that after the beginning of the industrial revolution, factor rents across the globe (but especially in England) tended to equalize to a world rate. With the expansion of overseas trade in the 18th and 19th century, the assumption of a closed economy becomes less palatable. In addition, the fundamental changes in the British economy make it difficult to believe the elasticity of substitution was the same before and after the industrial revolution. For all these reasons, I cut the sample at 1750, just before the beginning of the industrial revolution.

In Table 1, column 1, I estimate equation (3) using OLS, and find that the elasticity of substitution is 0.718, and significantly less than one. As mentioned earlier, this is a surprising result, since most contemporary estimates of the elasticity of substitution find that it is greater than one. In column 2, I add a time trend, and the estimate falls slightly to 0.695. Later when I relax the assumption of Hick-neutral technology growth, this time trend will take on economic significance, as will be discussed in section 3.

Figure 4 shows a residual plot of the regression in column 2 of Table 1. A residual plot is a useful way of showing the partial correlation between two variables in a multivariate regression, and is useful to see if the regression coefficient on a particular variable is driven by outliers. It is obtained by plotting the residuals of two regressions – one of the dependent variable on all the independent variables except for the independent variable of interest, and the other of the independent variable of interest of all other independent variables. The residuals of the first regression should contain all the variation in the dependent variable except for that due to the independent variable of interest, so plotting these against the residuals of the independent variable of interest netting out all other independent variables should provide a scatter plot of the partial correlation between the two variables. The relationship between the ratio of land rents to wages and population is positive and very

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5All OLS estimates use Newey-West standard errors with one lag since there is a large degree of autocorrelation in the pure OLS residuals.
tight. As a result, the estimate of the elasticity of substitution has an extremely small standard error. It is also interesting to note that over a 550-year span, the relationship remained very stable. One might expect that over the 550 year span of the sample something fundamental could have changed in the English economy regarding the importance of fixed resources and land – so it is remarkable to note that the linear relationship between the ratio of factor rents and population was almost exactly the same in 1750 as it was in 1200.

2.2 Instrument and 2SLS Results

There are two potential sources of bias in the above analysis. First, there is the issue of measurement error given the historical nature of the data. Estimates of medieval populations in general vary greatly, since they are usually based on a small subset of villages for which data exists and then extrapolated to the country as a whole. More accurate data on population based on family reconstruction are available only after 1541 (see Wrigley et al 1997). Since measurement error will cause attenuation bias to the coefficient on log population, this implies that its inverse will be biased upward. As a result, the true elasticity of substitution between land and labor will be even smaller than I estimate, strengthening my main qualitative finding that the elasticity of substitution between land and labor is significantly less than one.

Secondly, there is concern about the exogeneity of population changes. It is generally understood that factor quantities are endogenous to their prices. In this setting specifically, the Malthusian model implies that higher wages should lead to higher population growth, and low wages should lead to population decline. As a result, there could be a positive association between wages and population, independent of the effect which runs through the elasticity of substitution between land and labor.

Both statistical problems can be solved by using an instrument. In this paper, I use the timing of plagues in England as an instrument for population. In particular, I use the series of plagues known collectively as the Second Pandemic.

The Second Pandemic began in approximately 1346. It originated in Central Asia and ultimately affected the entire Afro-Eurasian continent. Although accurate estimates are difficult to obtain, it has been estimated that this first wave of the second Pandemic, commonly referred to as the Black Death, killed half of the inhabitants of China, a third of Europe, and an eighth of Africa. It was introduced into Europe in 1357 when Mongol armies besieging the Crimean city of Caffa catapulted infected corpses over the city wall in an attempt at biological warfare. The inhabitants of the city became infected, and the survivors fled to Sicily, from which the plague spread into the rest of Europe.
In England, the plague arrived shortly before 24 June, 1348 in Weymouth via a ship southwestern France. From Weymouth, the disease spread rapidly across the southwest of England, hitting Bristol first and reaching London in the fall. The spread of the disease slowed in the winter, but the next spring spread across all southern England and into the north. It reached York in May, and spread quickly over all of northern England during the rest of the summer, before dying out that winter. Although estimates disagree, the Black Death killed between 25% and 60% of the English population.

The first serious recurrence in England came in the years 1361-62, and then again in 1369, with a death rate of approximately 10-20% (Gottfried 1983). Over the following century the plague would return at intervals of five to twelve years, with continuously smaller mortality. There was a resurgence in severity between 1430 to 1480 – the outbreak in 1471 took as much as 10-15% of the population, while the death rate of the plague of 1479-80 could have been as high as 20% (Gottfried 1983). From 1480, the outbreaks decreased in frequency in England until the last great plague epidemic, the Great Plague of London in 1665-66. On the European continent, plagues continued to occur well into the 18th century.

There are three types of plague, which differ in the location of infection and vector of transmission. Each type of plague was present during the Black Death, and was caused by the same bacterium, *Yersinia pestis*. First is the bubonic plague, which affects the lymph nodes. It was transmitted via infected flea bites, and was carried by plague-resistant rats. Second was the pneumonic plague, which infected the lungs. It was transmitted through the air via tiny droplets of infected fluid. This form was especially deadly, since its main symptom was coughing and could be easily spread human to human. Last was the septicemic plague, which infects the bloodstream and is spread through direct contact with infected tissue or bodily fluids. Mortality from the plague was high, ranging between 30% to 95% of those infected.6

In Appendix A, I list every major national plague outbreak in England during the Second Pandemic, starting with the Black Death (1348-1350) and ending with the Great Plague of London (1665-1666). I can use the data in two ways to create an instrument: First, I can use an indicator variable for whether or not there was a national plague epidemic in that decade. Secondly, I can accumulate the number of plague outbreaks over time to create a “stock” of plague outbreaks in the last 100 years. My empirical results are similar using

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6Earlier I make that assumption that labor supply is a constant fraction of the population. One may worry that this may not be the case during the Black Death for two reasons. First, labor supply per person may fall due to morbidity caused by the plague. This does not seem to be the case, as the plague generally killed those infected within a week. Second, the fraction of the population working may change if the Black Death disproportionately affected certain age groups. This also does not seem to be the case – mortality rates for all age groups and social classes were similar. However, mortality rates were particularly high for some occupations, such as monks and priests who directly cared for the infected and dead.
either instrument.

It is important to note that I am *not* using the *magnitude* of plagues in terms of mortality as my instrument – I am using the *timing* of the plagues. It is easy to argue that when the plagues occur, the size of the mortality reduction is dependent on land-labor ratio: crowded locations should have higher rates of disease transmission, leading to higher deaths. In contrast, the key identifying assumption in this paper is that the *timing* of plague epidemics is exogenous to the land-labor ratio, and only affect relative factor prices through their effect on labor supply.

However, one may still worry that the timing of the plagues were endogenous to land scarcity, and therefore to its rental rate – which would imply failure of the exclusion restriction. At the center of this view is the idea that the plagues were a necessary and long overdue Malthusian correction of population, which had reached or even exceeded its sustainable level. Most historians, however, argue that is is not the case. (see Helleiner 1950, Hallam 1972, Herlihy 1997, Poos 1985, and Chavas and Bromley 2005. Hatcher and Bailey 2001 provides a further review.). They argue that the Black Death could not have been caused by a Malthusian crisis for several reasons. First, they argue that the halt in rapid population growth preceding the Black Death was due to weather and climatic shocks, not land scarcity. Before the Black Death, the population of England increased rapidly from 2 million in 1000 A.D. to 6 million in 1317, indicative of non-scarcity of land. In 1315-1322, England suffered a period of massive and generalized crop failures due to very extreme and rare weather conditions (Kershaw 1973), otherwise known as the Great Famine. In addition, in 1319 there was a severe cattle plagues which destroyed draught animals and lowered agricultural productivity. This famine led to increases in crime, displacement, and social upheaval, reducing population through the 1330s. They argue that without these weather shocks, population likely would have continued to grow rapidly, implying there was no Malthusian crisis.\(^7\)

Secondly, the timing of the plagues can be tied to specific events, such as a rat-laden ship arriving in a port. The precise ship which brought the first plague epidemic into England is well known. Later plague epidemics can be traced to similarly exogenous arrivals. Third, they argue that plague timing was truly random since they generally hit all of Europe simultaneously, even the countries whose populations had much different population dynamics and land-labor ratios than England. Figure 3 demonstrates that the spread of the Black Death throughout Europe was both rapid and virtually universal – and unlikely to be driven by land scarcity in a single country like England. The Black Death, they argue, would have

\(^7\)In fact, Campbell (2000) finds evidence that population did continue to increase in spite of the Great Hunger and its subsequent unrest.
been just as deadly and occurred at the same time whether or not there were population pressures. The fact that population dropped dramatically is not a proof that England was above a sustainable level of population. In fact, some very dense areas, such as Venice, Milan, Krakow, and some areas of southern France were mostly unaffected by the Black Death. Fourth, food prices were quite low on the eve of the Black Death, further indicating land was not scarce. Fifth, the ratio of land rents to wages was constant for the 150 years before the Black Death, whereas increasing population pressure would have implied a rising ratio.

Finally, the timing of plagues in England are not consistent with the idea that high populations cause plagues. If Malthusian pressures cause the plagues, then more plagues should occur when population is high, and less when population is low. In reality, we see the exact opposite happening – national plague epidemics occurred most frequent precisely when population was already at its lowest. After the Black Death, population remained low for 300 years, but plagues continued to occur, and in fact occurred more frequently. Appendix A lists all occurrences of plagues in England. Most of the plagues occurred in the 1400-1500s, even though population was already very low (see Figure 2), and became less frequent in the 1600s after population had risen in the late 1500s. In addition, Herlihy (1997) argues that the Black Death came too late if it were caused by population pressures, since population in 1348 was well off its peak. For all these reason, it is highly plausible that the timing of plagues is a source of exogenous variation in population, and hence the land-labor ratio.\(^8\)

In columns 3 and 4 of Table 1 are the results of the 2SLS regression of equation (3) with each instrument. When using the number of plagues in the last 100 years as an instrument, I estimate the elasticity of substitution between fixed and non-fixed factors to be 0.630. This is slightly smaller than the OLS estimate, which we should expect to be the case due to measurement error and the positive feedback between population and wages. Similarly, using the current plague indicator as an instrument, I find the elasticity of substitution to be 0.683. The F-test on the first stage instrument coefficient in each regression are 18.34 and 22.04 for plagues in the last 100 years and the plague indicator respectively.

Columns 5 and 6 repeat the analysis with a time trend. The results are similar, with the elasticity of substitution being 0.693 when I use the number of plagues in the last 100 years as the instrument, and 0.695 for the plague indicator. The first stage regressions are stronger in this case, with F-statistics of 126.29 and 33.2.

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8One may wonder whether repeated plague outbreaks were in fact the same outbreak, where the bacteria lie dormant in the population, and manifested itself when population pressure became large. However, this possibility is ruled out by the biology of the plague itself. In short, outbreaks tended to be short and intense, with mortality rates too high, and the duration and incubation period of the plague too short, to have been dormant in the population.
2.3 Robustness

Up to this point, I have used a simple model to estimate the elasticity of substitution. However, one may worry about the validity of several key assumptions. This section will discuss the direction of bias in my estimates when relaxing these assumptions, as well as present robustness checks.

Fixed Natural Resources

I assume that the total endowment of natural resources, $X_t$, is constant over the entire sample. There are two possible issues with this assumption. First, one may worry that population pressure induced land use to change: for example, land not in use due to its poor quality may be transformed into productive agricultural land, or land already in use in one agricultural sector may be converted into another. Since Clark’s land rent series is not solely based on cropland rents, but rather is a national rent index over many different types of land uses, (such as arable land, meadow, marsh, pasture, or mixed use), changes in land use between land already used in production should not cause a problem.

Second, the overall amount of land available for use in any type of production may be changing due to land reclamation, soil depletion, or erosion. If the amount of available land increased over the sample period, this could bias our estimates. At the very extreme, so long as national borders don’t change, the overall amount of land available for use in any type of production remains the same over time, so the bias caused by land reclamation must be bounded. However, there is good evidence that land has been reclaimed in England over time (such as the draining of the Fens in eastern England), so it will be useful to think about the bias this would cause in our results.

Assume that rising population densities increases the amount of land used in the production of goods. This implies a positive correlation between $P_t$ and $X_t$, meaning the land/labor ratio would change less in response to population growth than if $X_t$ were fixed. Therefore, the coefficient on $P_t$ in regression (3) would be underestimated under the assumption of a fixed $X$. Since $\sigma$ is the inverse of the coefficient on $P_t$, the estimated elasticity would be too large. Therefore, although changing $X_t$ may bias my estimate of $\sigma$, it actually strengthens my qualitative argument that the elasticity is less than one.

The opposite would be true if resource depletion were an issue. If resource depletion implied a negative correlation between $P_t$ and $X_t$, then the estimated elasticity would be too small. My qualitative result depends on which effect was larger, the land reclamation effect, or the land depletion effect. As stated above, the evidence is suggestive that land reclamation was the dominant effect in pre-industrial England, and therefore my estimates
for $\sigma$ are likely to be too large – strengthening my argument that the elasticity was less than one.

**Labor as the only Non-fixed Factor**

I assume that the only factors of production are land and labor in this economy. I can easily extend the model to account for physical capital by allowing $N_t$ in equation (1) to become a capital-labor composite factor, $N_t = K_t^{\delta}L_t^{1-\delta}$. From CES production function, I derive the following relationship between $L$ and $K$:

$$K = \frac{\delta}{1-\delta} L \frac{r_L}{r_K}$$

Substituting this into $N_t$ above results in the following regression equation:

$$\ln \left( \frac{r_X}{r_t} \right) = \alpha + \frac{1}{\sigma} \ln (P_t) + \zeta_1 \ln (w_t) + \zeta_2 \ln \left( \frac{r_K}{r_t} \right).$$

(4)

This is the same as equation (3), without assuming $\zeta_1 = 1$, and controlling for the wage and rental rate on capital in the regression. By extension, I can control for the presence of any non-fixed factor in the composite factor $N_t$ by simply controlling for its rate of return.

Clark (2001, 2002, 2007b) provides data on the rental rate of capital $r_K$ in addition to the data on wages and land rents. He uses documents transferring property by gift or sale to a religious house – or cartularies – which include information on rental payments to capital to calculate capital’s rate of return. Using this additional series of data, I estimate equation (4) in table 3. Column 1 reports a simple OLS regression with a time trend but without controlling for capital. Columns 2 and 3 are 2SLS estimates of the same regression using each of the two instruments. The results are similar to the simple OLS regression obtained in table 1, and are shown primarily for comparison.

Columns 4-6 repeat the previous exercise while controlling for the return on capital. The estimated elasticity rises from about 0.6 to 0.8, depending on the specification. While the elasticity is still less than one, I can no longer reject that it is equal to one. This is problematic for my qualitative conclusion in the previous sections that the elasticity of substitution is less than one. However, as we will see in the next section, when I control for factor induced technical change, the estimated elasticity of substitution will fall back to its previous level of about 0.6, and be significantly less than one (see columns 7-9).

Human capital is omitted from the model for two reasons. First, low levels of human capital before the industrial revolution suggest it likely wasn’t an important factor of production. Literacy rates for males in England reached 25% only after 1600, and were even
lower for females. Even during the first phase of the industrial revolution, human capital had a limited role in the production process since factory work did not require literacy (see Galor 2005). For example, the fraction of children 5-14 enrolled in primary education did not exceed 10% until after 1850 (Flora et al 1983). Second, on a more practical level, data on human capital over this period are difficult, if not impossible, to obtain.

**Labor as a Constant Fraction of Population**

I assume that labor is a fixed fraction of the population. However, it is possible that labor force participation rates changed as a result of the Black Death. For example, high wages after the Black Death may have induced more labor into the market. Many scholars, including Voigtländer and Voth (2013b) note that female labor force participation increased after the Black Death in response to higher wages, labor scarcity, and the change in production from crops to pastoral goods.

Just as before, we can determine the direction of bias if labor supply was not a constant fraction of population. Suppose that labor supply per person increased when population was low and decreased when population was high. This implies that fluctuations in population overstate the true change in labor supply, and therefore the coefficient on population will be underestimated. This implies \( \sigma \) will be overestimated – further strengthening my finding that the elasticity of substitution between land and labor is less than unity.

### 3 Biased Technological Progress and Induced Innovation

I now consider the following CES production function with factor-specific technologies \( A^X \) and \( A^N \):

\[
Y_t = \left[ (A^X_t X)^{\rho} + (A^N_t N)^{\rho} \right]^{\frac{1}{\rho}},
\]

(5)

This is the same as (1), except now productivity is factor specific. Following the methods in Section 2, this results in the following regression equation:

\[
\ln \left( \frac{r_t}{w_t} \right) = \alpha - \frac{1}{\sigma} \ln \left( \frac{A^X_t}{A^L_t} \right) + \frac{1}{\sigma} \ln (P_t) .
\]

(6)

In addition to difficulties estimating the regression model as mentioned in section 2, we now have \( A^X_t \) and \( A^L_t \), which will affect the factor rent ratio and are unobserved. But by making a few parsimonious modeling assumptions, I can attempt to control for factor-induced productivity growth.
Assume that the growth rates of $A_X$ and $A_L$ ($g_X$ and $g_L$ respectively) vary depending on the land/labor ratio. Since $X_t$ is fixed, population size is a sufficient statistic for the land-labor ratio. Define the difference between the land- and labor-specific growth rates to be $g(P_t) = g_X(P_t) - g_L(P_t)$. From the theory of induced innovation, assume that when land is scare (i.e. when population is high), $g_X$ will be high and $g_L$ will be low, implying $g'(P_t) > 0$. The specific functional form of $g(P_t)$ is unknown. In my estimation, I will use a simple linear specification, such that $g_t = \gamma + \theta P_t$.

It can be shown that this implies a new regression equation (see Appendix B for details):

$$\Delta \ln \left( \frac{r_t}{w_t} \right) = \beta_1 + \beta_2 P_t + \beta_3 \Delta \ln (P_t) , \quad (7)$$

Equation (11) states that the change in the ratio of factor rents is affected not only by the change in the land-labor ratio, but also the level of the land-labor ratio $P_t$ itself via its effect on $\frac{A_X}{A_L}$ through induced innovation.9 The intuition is fairly simple: The level of population will cause a change in the growth rate of factor prices since if there is population pressure, there will be an incentive to innovation around the scare resource – land. As a result, $A_X$ will grow faster than $A_L$, causing land rents to grow faster than wages. This is in addition to the direct effect of the land-labor ratio on the ratio of factor prices as dictated by the elasticity of substitution. Hence from the coefficient on the level of population we can recover the magnitude of induced innovation, while we identify the elasticity of substitution from the coefficient on the growth rate of population. In terms of the parameters, we identify $\gamma$, $\theta$, and $\sigma$ as $\gamma = -\frac{\beta_1}{\beta_3}$, $\theta = -\frac{\beta_2}{\beta_3}$ and $\sigma = \frac{1}{\beta_3}$.

I estimate equation (11) in Table 2. Column 1 repeats the simple OLS regression from Table 1 with a time trend and controlling for induced innovation. The elasticity of substitution is about 2/3, almost identical to the estimates in Table 1. When using 2SLS with each instrument, the elasticity becomes smaller but qualitatively similar to the OLS regression.

The parameters $\gamma$ and $\theta$ jointly determine the degree of biased technological progress, or $g_X - g_L$. Specifically, $\gamma$ estimates the growth rate of $\frac{A_X}{A_N}$ if population were near zero. In

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9 A slightly different functional form for the growth rate, $g(P_t) = \gamma + \theta \ln(P_t)$, would lead to equation (9) becoming $\Delta \ln \left( \frac{r_t}{w_t} \right) = \beta_1 + \beta_2 \ln(P_t) + \beta_3 \Delta \ln (P_t)$. In this case, both of the regressors contain $\ln(P_t)$, so to estimate the effects separately the regression equation would need to be rewritten as $\Delta \ln \left( \frac{r_t}{w_t} \right) = \beta_1 + (\beta_2 + \beta_3) \ln(P_t) + \beta_3 \ln(P_{t-1})$. Doing so does not materially affect my findings, and results of this estimation can be provided upon request.

10 In the simple regression model without controlling for induced innovation, I used plague deaths as an instrument for population. When I include induced innovation in equation (7), population now enters the equation twice – both in the level of population and in the growth rate. In order to correctly identify the parameters, I now use two instruments – plagues for the current level of $P_t$ and change in plagues for the change in $\ln(P_t)$. In the results that follow, I will therefore report two F-stats for the first stage regression, corresponding to the joint significance of the instruments in each of the first stage regressions.
the OLS regression in column 1, I estimate that $\gamma$ is -0.163%, implying that technological progress would be labor saving in this case. This is intuitive – if population is low, labor would be scarce, and innovation would be labor saving. For the 2SLS regression in columns 2 and 3, the estimates are -0.302% and -0.361%, or approximately double the OLS coefficient.

The parameter $\theta$ represents the degree of induced innovation – the extent to which the level of population influences the growth rate of the ratio of factor prices. A positive coefficient implies that higher populations lead to faster growth in $\frac{A_X}{A_N}$. In column 1 (the OLS regression), I estimate that $\theta$ was 0.051%. This means that for each million people in England, $g_X - g_N$ was 0.051% higher. When I instrument for population in columns 2 and 3, the estimate approximately doubles to about 0.1% per year.

Since the overall growth rate of $\frac{A_X}{A_N}$ is equal to $\gamma + \theta P_t$, we can calculate the population at which $g_X$ and $g_N$ grow at the same rate. This level of population will be such that $P = -\frac{\gamma}{\theta}$. If population is higher than this value, induced innovation will lead to relatively to be land saving innovations, and below this value technology will be relatively labor saving. In each of the regression in columns 1-3, I calculate this “direction-neutral” level of population to consistently be between 3.2-3.4 million inhabitants.

Finally, I re-estimate the augmented model with capital to allow for induced innovation in columns 7-9 of Table 3. Column 7 shows the OLS result controlling for technical change, while columns 8-9 show the estimates with each of the two instruments. The elasticity of substitution is slightly lower than before, from 0.583 in the OLS specification to 0.429 in one of the 2SLS regressions. The amount of implied induced innovation is the exact same in the two OLS equations at 0.051% per million people. However, in the 2SLS regression, the degree of induced innovation is slightly smaller when capital is included, down to 0.077% from 0.090%.

4 Implications of the Elasticity of Substitution

So why is the finding that the elasticity of substitution between fixed and non-fixed factors is less than one significant? The degree to which factor prices change as their quantities change, as measured by the elasticity of substitution, greatly effect the dynamics and steady states of a large number of economic models.

Under diminishing marginal returns, the marginal product of any factor of production must fall as the quantity of that factor accumulates. The elasticity of substitution between factors influences the speed by which this marginal product falls as the factor is added. To demonstrate this, consider the following CES production function with Hicks-neutral technological progress:
\[ Y_t = A_t (\alpha L_t^\rho + (1 - \alpha)X^\rho)^{\frac{1}{\rho}} \]

where \( L_t \) is labor at time \( t \), and \( X \) is a fixed factor. In this case, the elasticity of the wage with respect to labor is

\[ \frac{\partial w_t}{dL_t} \cdot \frac{L_t}{w_t} = -\frac{\phi_{x,t}}{\sigma}, \]

where \( \phi_{x,t} \) is the fraction of income paid to land. This elasticity is negative and positively related to \( \sigma \) – in other words, as more labor is added, the wage will fall faster if the elasticity of substitution is low.

In this section, I will discuss how this elasticity effects two general types of models where population matters: models of the transition from a Malthusian economy to a modern economy with sustained economic growth (e.g. unified growth theory models) and models of the effect of population on economic development in the developing world.

4.1 Malthusian Models

Recently, many economists have been interested in Malthusian economies and their dynamics. Lee (1973) was among the first to estimate a model of dynamics around the Malthusian steady state. More recently, Ashraf and Galor (2011) develop a model of the Malthusian economy and test the model’s predictions using historical data on income and population size. There is also a large and active literature on understanding the transition from the Malthusian steady state to modern growth, as well as the causes of the Great Divergence.

In this section, I show how the elasticity of substitution between land and labor affects several types of these Malthusian models. I group the literature on the takeoff to growth from Malthusian stagnation into three categories, and discuss how this elasticity affects each type of model. I resimulate an original model in each category to demonstrate how the results of the model depend on the the elasticity of substitution.

Models of Biased Technological Progress

Hansen and Prescott (2002) present a unified growth model in which biased technological progress causes the demographic transition. In their model, there are two sectors: the Malthus (agricultural) sector which uses land, labor, and capital as inputs; and the Solow (manufacturing) sector which only uses labor and capital. Each sector produces the same good, and technological progress is significantly faster in the Solow sector than Malthus sector. Initially, the Malthus sector is the only sector operative, and the economy is in a...
Malthusian steady state.

In this setup, the escape from the Malthusian steady state occurs when technology in the Solow sector grows to the point that it is profitable to produce output. When both sectors are operative, capital and labor begin to flow from the Malthus sector to the Solow sector until virtually all the capital and labor are employed in the Solow sector. Once this occurs, the economy behaves according to the Solow model where technological progress translates into higher income per capita rather than higher population size. The transition is inevitable and rapid in their model, since the rate of technological progress in the Solow sector is over 16 times faster than in the Malthus sector.

To assess the effect of $\sigma$ in Hansen and Prescott’s model, I allow for different elasticities of substitution in the Malthus sector.\textsuperscript{11} The only other change I make to their model is to slow the model down by assuming that each model period is one year, rather than 35 years as in the original model. All the parameters in the model are adjusted accordingly. I do this because in the original model, the transition happens so quickly (for example, approximately 80% of the factors in the economy move into the Solow sector in one model period in the original model), that it would be hard to effectively show how the elasticity affects the transition speed of the model with such a fast transition.

I consider three cases for $\sigma$: 1 as in the original Hansen and Prescott model, compared with 0.5 and 2.\textsuperscript{12} Figure 5 shows the speed by which capital flows out of the Malthus sector into the Solow sector under each of these three cases. I find that the lower the elasticity of substitution, the slower factors leave the Malthus sector once the Solow sector becomes operative. This result is intuitive – as capital and labor leave the Malthus sector, their marginal products rise faster if the elasticity is low, causing more capital and labor stay in the Malthus sector. In the first period of the model, there is more than twice as much capital and labor in the Malthus sector when $\sigma = 0.5$ compared with when $\sigma = 2$. Over time, this ratio falls, but there always remains more factors in the Malthus sector when $\sigma$ is high. For example, when $t = 50$, the Malthus sector has approximately 22% less capital and labor when $\sigma = 2$ as opposed to when $\sigma = 1$.

Models that Depend on Population Size

One set of models explaining the transition to growth from Malthusian stagnation feature a positive relationship between the rate of technological progress and population size (Kremer 1993, Diamond 1997, and Galor and Weil 2000, among others). In many of these models,

\textsuperscript{11}In their original model, Hansen and Prescott used Cobb-Douglas production functions in both the Malthus and Solow sectors, which implicitly assumes a unit elasticity of substitution.

\textsuperscript{12}The full analysis and simulation results are too long to be outlined in this paper, but is available upon request.
once technological progress is fast enough, the economy begins to transition away from the Malthusian steady state. Since the rate of technological progress is determined by the size of the population, the transition begins once population reaches a threshold level in many of these models.  

One of the surprising results from these models is that improvements in technology will lead to smaller increases in population if the elasticity of substitution is low. To see this, consider a technological advancement which raises the productivity of labor, and hence wages, by some fixed percentage in a Malthusian economy. Higher wages will increase population size either by increasing fertility or reducing mortality, which will in turn reduce wages back down to subsistence. Since a low elasticity of substitution causes wages to fall faster with an increase in labor, a given increase in the wage will require a smaller increase in population to return back to subsistence.

As a result, productivity must rise by a larger amount to achieve the threshold level of technology required for the takeoff to sustained growth, since each unit change in technology increases population by a smaller amount. The empirical prediction would be that, given similar rates of technological progress, societies which have lower elasticities of substitution between land and labor (perhaps because of different crop availability or geographic characteristics) would emerge from Malthusian stagnation later.

In addition, a second effect of the elasticity of substitution on the growth rate of technological progress would exacerbate this effect. Technological progress should be slower at every point in time in the economy with a smaller elasticity of substitution, because the level of population will be lower. To see this, consider two economies identical in every way except for their $\sigma$. In the first period, both countries experience a similar positive technological shock, increasing wages in both countries by the same amount. In the next period, population in both countries will rise, but it will rise faster in the country with a high elasticity of substitution as explained above. Since technological progress is positively associated with population size, technological progress should then be faster in the second period in the country with high $\sigma$, and will continue to be divergently higher as time progresses.

To show the effect of this elasticity on the timing of the transition to sustained growth, I modify Lagerlof’s (2006) simulation of Galor and Weil (2000). Due to space constraints, a full description of Lagerlof’s model will not be included in this paper, but the full simulation results are available upon request. The simulation is only modified from the original in the elasticity of substitution between land and labor. Specifically, I simulate the model using

\[ \sigma \]

As mentioned in Lagerlof (2006), a higher time cost of children ($\tau$) reduces the amount of oscillation in the model caused by population booms and busts. Since an analysis of the oscillations is not the purpose of this exercise, and since it was not a part of the original Galor and Weil model, I also modify the fixed time
values of $\sigma$ of 0.5, 1, and 2. The results are shown in Figure 6. Just as predicted, a higher elasticity of substitution leads to a shorter time to transition to sustained growth. In fact, moving from an elasticity of substitution of 2 to an elasticity of 0.5 approximately doubles the time to transition from about 25 model periods to 45 model periods.

**Models that Depend on Wage Levels**

Another set of models which depend on the elasticity of substitution are those which depend on achieving some threshold level of income to transition. An example of a model in this category are Voigtländer and Voth (2006), where the probability of transitioning to growth depends positively on the level of income. Other models depend on rising incomes to spur demand for the manufacturing sector or other luxury goods, thereby igniting the industrial revolution (for an example, see Stokey 2001).

In the classical Malthusian model, when positive technology shocks occur, the subsequent increase in income per capita will only be fleeting, since future population growth will eventually lower wages back to the Malthusian steady state. The speed by which income falls back to the steady state is function of the elasticity of substitution between land and labor – the faster wages fall with increased population, the faster income per capita falls as well. However, the converse is also true. If there is a reduction in population due to wars, plagues, etc., then incomes will rise higher as a result of this loss if the elasticity of substitution is low.

Voigtländer and Voth (2013a) explain persistent income differences using a model in which positive wage shocks are reinforced by increases the amount of plagues, wars, and the rate of urbanization, eventually leading to perpetually higher death rates and permanently higher incomes. In addition, Voigtländer and Voth (2013b) argue that Western Europe was able to maintain higher levels of pre-industrial income than the rest of the world due to unique fertility patterns from delayed marriage which led to smaller populations. As population was kept low and wages remained high, demand for industrial goods increased, leading to the industrial revolution in Europe. If the elasticity of substitution in pre-industrial England was small, then the positive effect of the Black Death on wages would have been higher. Therefore, the reinforcing effects of plagues, wars, urbanization, and European marriage patterns would be even stronger, and the eventual divergence in European incomes just that much higher. Presumably, this would cause the industrial revolution to happen even faster.

But even aside from this type of model in which pre-industrial income levels determine
when countries will industrialize, divergence in pre-industrial incomes is interesting in its own right. A standard Malthusian prediction is that income levels will return to subsistence in the long run, and hence there should be no persistent differences in cross-country incomes in the Malthusian world unless there are differences in the subsistence level of income. However, Vollrath (2011) finds that a larger share of income paid to land can lead to higher income levels before the transition to modern growth using a simple 2-sector model. The key to his result is that agricultural production processes which have a high labor intensity reduce the speed by which increase population reduces wages, allowing for a higher share of labor to remain in agriculture instead of transitioning to the industrial sector. However, from (8) we learn that not only does labor intensity (as measured by the fraction of output paid to the factor) reduce the speed by which the wage falls as labor is added, but also the elasticity of substitution between land and labor as well. As a result, it follows that the higher the elasticity of substitution between land and labor, the smaller the decline in the real wage due to population growth, and the lower the level per capita income before the transition.\footnote{A full model closely following Vollrath (2011) but with a non-unit variable elasticity of substitution was derived by the author to confirm this result, which will not be available here due to space constraints, but is available upon request.}

Interestingly, the predicted effect of the elasticity of substitution between land and labor on the timing of industrialization is the opposite in these “income level” models as opposed to the previous two types. As discussed above, Galor and Weil (2000) and Hansen and Prescott (2002) predict that higher elasticities of substitution should cause industrialization to happen faster, not slower as in Voigtländer and Voth (2006) and Vollrath (2011). It would be interesting therefore to compare the elasticity of substitution across crop mixes in the pre-industrial era in different regions of the world, and test whether high elasticity regions transitioned faster or slower than low elasticity regions. Unfortunately, high quality estimates of land rents extending earlier than 1700 are unavailable outside of a few countries in Europe, making such a comparison impossible with current data.

### 4.2 Population and Development

Second, the extent to which population levels affect output is a very old question within economics. It is generally agreed that the direct effect of population on income is negative in the presence of a fixed factor. However, there is significant disagreement on the degree of substitutability of between factors. For example, Boserup (1965) maintained that crop intensification would occur in the face of population pressure, leading to proportional increases in food production. This is just another way of saying the elasticity of substitution between fixed and non-fixed factors is high. On the other hand, Ehrlich (1968) maintained
that global food production could not increase much more beyond the level it was when his book was written without increases in the area under cultivation, which is just another way of saying that the elasticity of substitution must be low.

It is clear that larger populations cause resources per capita to shrink. If the elasticity of substitution is low, additional labor does little to increase production, leading to lower average products and smaller incomes per capita. This implies population size will have a larger negative effect on income levels. Therefore, models which predict the effect of population size on output must be effected by this elasticity. In addition, the efficacy of programs intending to raise income per capita, but which also change population size (such as disease eradications which lower mortality) will depend on this elasticity.

Surprisingly little concrete empirical research has been devoted to this most simple, yet important question. Most studies of the effect of population on output deal with effect of the population growth rate, rather than population level. For example, the effects of dependency ratios/demographic structure, accumulable factor dilution, changes in fertility and mortality and their effect on labor force participation, savings, etc. and so forth all work through the growth rate of population rather than its absolute size. This is even more surprising when one considers that the father of population economics, Malthus, originally framed the question in terms of levels, not rates. He conjectured that in the face of a fixed amount of arable land, food production would not be able to increase fast enough to sustain an exponentially growing population with a shrinking land-labor ratio. This argument contains an implicit assumption about the substitutability of land in food production – that it is small.

Over the last 200 years, the dire predictions of Malthusian pessimists such as Ehrlich have not come to fruition. However, this does not imply than large populations do not depress wages. Recently, Weil and Wilde (2009) calculate the effect of a doubling in population size on income per capita. Assuming that each factor of production is paid its marginal product, one can solve for the fraction of national income paid to each factor, and then take the ratio of incomes in two points in time based on differing levels of inputs. Using a CES production function similar to (5), they derive

\[
\frac{y_t}{y_0} = \left( 1 - \phi_0 \right) + \phi_0 \left( \frac{L_0}{L_t} \right)^{\frac{\sigma - 1}{\sigma}} \left( \frac{\sigma}{\sigma - 1} \right),
\]

where \( y \) is income per capita, \( \phi \) is the fraction of national income paid to land, and the subscripts index time. In this simple example, all that is needed to estimate the effect of population size on income per capita is data on \( \phi \) and \( \sigma \).\(^{16}\) Taking several values of each

\(^{16}\) \( L_0/L_t = 0.5 \) since Weil and Wilde consider a doubling in population.
parameter and shows the corresponding change in income per capita, Weil and Wilde show
the larger the elasticity of substitution between land and other factors, the smaller is the
impact of an increase in population on standards of living.

Another recent paper which depends crucially on the elasticity of land and other factors
is Ashraf et al (2008). Using a simulation model, they estimate of the effect of an unspecified
health intervention which improves life expectancy from 40 to 60. In addition to improving
the productivity of workers via health and educational human capital, this reduction in
mortality increases population size.

Using their model, Figure 7 shows the effect of this intervention on income over a 165 year
time horizon. With an elasticity of substitution greater than unity, the health intervention
improves income in the long run. In this case, the negative effects of population size on
income are small relative to the productivity benefits from the health intervention. However,
the overall effect on income becomes negative if the elasticity becomes small enough. In this
case, since it is difficult to substitute away from land, shrinking land per person has a large
negative effect on production. This negative effect of population on income overwhelms the
positive effect of worker productivity. In the case of an elasticity of substitution of 0.5,
income per worker decreases by about 25% after 50 years and never recovers. Thus the
same health intervention could have completely opposite results depending on the elasticity
of substitution.

Finally, Ashraf et al (2011) use a similar model to quantify the effect of a reduction
in fertility on income per capita. They model several different channels by which fertility
reduction affects income, including what they call the “Malthus” channel, or the effect on
income per capita due to fixed factor congestion. In the long run, the elasticity of substitution
plays a large role in determining the magnitude of the effect on income per capita. Figure 8
shows the effect of this intervention on income over a 165 year time horizon. If $\sigma = 2$, the
Malthus effect is small since land is easily substitutable in production. Income 150 years
after the fertility intervention is about 45% higher. However, if $\sigma = 0.5$, income rises to
about 130% higher than baseline over the same time horizon – about three times higher
than the previous case. In the long run, the Malthus channel becomes the most important
channel by which fertility decline affects income.

These studies show that understanding the value of $\sigma$ is essential for accurately assessing
the impact of policies which affect population. The extent to which fixed factors can be
substituted for non-fixed factors in production, especially in agriculture, is central to
understanding population’s affect on living standards. Considering the large investment of

17 As with the other models presented in this paper, the full simulation results will not be presented due
to space constraints, but is available upon request.
resources devoted to health interventions and promoting fertility control across the developing world, the fact that almost no research until now has been devoted to the estimation of $\sigma$ is quite surprising.

5 Conclusion

The elasticity of substitution between fixed and non-fixed factors is an extremely important parameter in a large number of economic models. It governs dynamics around the steady state in Malthusian models of the economy. It also is essential for understanding the movement away from the Malthusian steady state and takeoff to modern growth which has occurred in most areas of the world during the last 200 years. It predicts speed of the demographic transition, the order in which countries will industrialize, and the size of pre-industrial divergence in income levels. In addition, it has important applications to the question of how population size affects economic development. The extent to which fixed factors can be substituted in production has implications for how strong the negative effect of population on growth is. The efficacy of interventions which affect population size, such as family planning programs or health interventions which affect mortality, will be affected by this substitutability.

Despite the importance of this parameter, no credible estimates of this elasticity are available. In this paper, I estimate this elasticity in pre-industrial England, and find that the elasticity of substitution was about 0.6 – significantly lower than most models assume. In addition, I estimate the direction of biased technological progress and the degree of induced innovation. I find that technology was scarce-factor saving. After the Black Death when land was plentiful and labor scarce, labor productivity growth grew faster. Later when population grew again and land became scarce, land productivity grew faster. An additional 1 million people (an increase of about 1/3 over population’s median level over the period 1200-1750) increased the differential growth rate between land- and labor-augmenting productivity growth by about 0.1% per year.

Finally, I discuss how the results of several categories of models of the demographic transition would change if the elasticity of substitution were smaller. The speed of the demographic transition in Hansen and Prescott (2002) would be significantly slower if the elasticity of substitution were consistent with my estimates. In addition, countries with low elasticities of substitution would takeoff later to sustained economic growth under models such and Galor and Weil (2000). This leads to a testable implication of Galor and Weil’s model – countries with geographic advantages in growing crops which have high land-labor substitutability should industrialize first. In addition, a small elasticity of substitution should
lead to higher levels of pre-industrial income per capita, similar to the finding of Vollrath (2011).

One shortcoming of the current study is the issue of external validity. It would be highly useful to identify the correct elasticity of substitution between land and labor in a country where population is rapidly expanding, such as sub-Saharan Africa. The extent to which labor might substitute for natural resources has implications for the efficacy of policies which aim to reduce population, such as family planning programs. The higher the land-labor substitutability, the less effective these programs will be. It also has implications for health interventions which may prevent mortality. For example, a large amount of international aid recently has been spent to disseminate bednets throughout Sub-Saharan Africa. This will likely reduce morbidity and increase health human capital in this region, leading to productivity gains. But these productivity gains may be undone by the reduction in wages which may occur from the increased population size. Which effect will dominate depends on the elasticity of substitution between land and labor. While pre-industrial England provides the ideal setting to estimate the appropriate elasticity of substitution to use in Malthusian growth models, it may be inappropriate for Sub-Saharan Africa.

References


Figure 3: Spread of the Black Death in Europe: 1347-1351

Pestis Prima -- the first wave of plague -- moved very quickly along sea and land trade routes.

Figure 4: Partial Correlation Residual Plot -- Ln(Land Rent/Wages) on Ln(Population)
Figure 5: Capital in the Malthus Sector under Different $\sigma$

Figure 6: Lagerlof (2006) (Galor and Weil 2000) with different EOS
Figure 7: Ashraf, Lester, and Weil (2008)
Effect of Health Intervention on Income per Capita

Figure 8: Ashraf, Weil, and Wilde
Effect of Decrease in TFR of 1.00 on Income per Capita
Land Share of Income = 30%
No Labor Force Participation or Schooling Effects
### Table 2: Results with Biased Technology Change

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<td>2SLS Plague Current</td>
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<td>Δln(r/w)</td>
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*** p<0.01, ** p<0.05, * p<0.1. Standard errors in parenthesis. OLS regressions use Newey-West errors with one lag. The significance levels on Δln(Pop) and Implied σ test whether the coefficient is different from one. All other significance levels test difference from zero. Since two instruments are used in the 2SLS regressions, the two 1st stage F-tests values report the joint significance of the two instruments in each of the two 1st stage regressions.
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*** p<0.01, ** p<0.05, * p<0.1. Standard errors in parentheses. OLS regressions use Newey-West errors with one lag. The significance levels on ln(Pop), ln(ln(Pop)), and Implied σ test whether the coefficient is different from one. All other significance levels test whether they are different from zero. When two instruments are used in the 2SLS regressions, the two 1st stage F-tests values report the joint significance of the two instruments in each of the two 1st stage regressions.
Appendix A: Plague Epidemics

List of national epidemics of the plague. List compiled by Brian Williams, 1996.

1348-1349  The Black Death.

1361  Pestis secunda or Pestis puerorum [Gottfried (1983) pg. 130]; [Shrewsbury (1970) pg. 23].

1369  Pestis tertia [Shrewsbury (1970) pg. 23].

1375  Pestis quarta [Shrewsbury (1970) pg. 23, 135-136].

1379-1383  Pestis quinta; 'In 1379 there was a great plague in the Northern parts...under the year 1382, a very pestilential fever in many parts of the country' [Creighton (1965) pg. 218]; London was afflicted in 1382, with Kent and others parts in 1383 [219].


1399-1400  [Gottfried (1983) pg. 131].


1411-1412  'another national epidemic' [Gottfried (1983) pg. 131].

1420-1423  Norfolk, 'but the Rolls of Parliament bear undoubted witness to a very severe prevalence of plague in the North about the same time' [Creighton (1965) pg. 221]; 1420 and 1423 [Gottfried (1978) pg. 36]; 1423 [Gottfried (1983) pg. 131].


1433-1435  'Here then, early in 1434, is the first distinct suggestion in the period 1430-1480 of something more that a local or regional epidemic' [Gottfried (1978) pg. 37]; 'a national epidemic that lasted from 1433 to 1435' [Gottfried (1983) pg. 132]; 1433 or 1434 [Shrewsbury (1970) pg. 143].


1463-1465 [Creighton (1965) pg. 229]; 'From 1463 to 1465, another severe epidemic hit the entire kingdom' [Gottfried (1983) pg. 132]; 1463 "a greate pestilence...all England over" [Shrewsbury (1970) pg. 146].

1467 'In 1467 another epidemic swept through parts of England, and was possibly national in scope. If the Rolls of Parliament are to be believed, it was unquestionably an epidemic of plague' [Gottfried (1978) pg. 42]; [Gottfried (1983) pg. 132].

1471 'evidence indicates that this epidemic was one of plague' [Gottfried (1978) pg. 44]; 'in 1471, all of England was overwhelmed' [Gottfried (1983) pg. 132].

1479-1480 'This year [1479] saw great mortality and death in London and many other parts of this realm' [Creighton (1965) pg. 231-232]; 'the great epidemic of 1479 in London and elsewhere' [286]; 'The most virulent epidemic of the fifteenth century was the plague of 1479-1480' [Gottfried (1978) pg. 14]; 'From autumn to autumn, a combined epidemic of bubonic and pneumonic plague devastated all of Britain' [Gottfried (1983) pg. 133].

1499-1500 'the great epidemic of 1499-1500, in London and apparently also in the country' [Creighton (1965) pg. 287]; [Gottfried (1978) pg. 14]; [Gottfried (1983) pg. 156]; the sixteenth century opened with 'a great pestilence throughout all England' [Shrewsbury (1970) pg. 159].

1509-1510 [Gottfried (1978) pg. 156]; 1509, a 'great plague' that afflicted various parts of England [Shrewsbury (1970) pg. 160].

1516-1517 [Gottfried (1983) pg. 156].

1523 [Shrewsbury (1970) pg. 163].

1527-1530 [Gottfried (1983) pg. 156].

1532 'There is supporting evidence that the disease was widespread' [Shrewsbury (1970) pg. 168].
1544-1546 'several scattered, localized outbreaks of plague in England' [Shrewsbury (1970) pg. 178], 1545 north-east [pg. 180], south coast [pg. 181], 1546 westwards [pg. 182].

1563 'probably the worst of the great metropolitan epidemics' [Shrewsbury (1970) pg. 176], 'and then extended as a major national outbreak of it' [Shrewsbury (1970) pg. 189].

1585-1587 'bubonic plague was busy in numerous places in England in the years from 1585 to 1587 inclusively' [Shrewsbury (1970) pg. 237].

1593 the 'great metropolitan and national epidemic of 1593' [Shrewsbury (1970) pgs. 176, 222].

1603-1604 [Shrewsbury (1970) pg. 264].

1609-1610 'The next two years, 1609 and 1610, witnessed several severe outbreaks of bubonic plague in English towns' [Shrewsbury (1970) pg. 299].

1625 'the great outburst of 1625' [Shrewsbury (1970) pg. 313].

1637 'widely distributed in 1637 and a number of places experienced more or less severe visitations of it' [Shrewsbury (1970) pg. 389].

1645 'The year 1645 was one of severe plague in several towns at the same time' [Creighton (1965) pg. 557].

1665 The Great Plague, affecting London in the main.

Appendix B: Regression with Induced Innovation

Assume that the growth rates of $A_X$ and $A_L$ ($g_X$ and $g_L$ respectively) vary depending on the land/labor ratio, such that when land is scarce (i.e. when population is high), $g_X$ will be high and $g_L$ will be low. We can rewrite $\ln \left( \frac{A_X}{A_L} \right)$ as:

$$\ln \left( \frac{A_X}{A_L} \right) = \ln \left[ \frac{A_{X_{t-1}} \left(1 + g_X^t(P_t)\right)}{A_{L_{t-1}} \left(1 + g_L^t(P_t)\right)} \right] = \ln \left[ \frac{A_{X_{t-1}}}{A_{L_{t-1}}} \left(1 + g_t(P_t)\right) \right],$$
where \( g_t = \frac{(1 + g_X(P_t))}{(1 + g_L(P_t))} - 1, \) \( \frac{\partial g_t}{\partial P_t} > 0. \) By iterating this equation and using the approximation that \( \ln (1 + x) \approx x, \) we can express the level of \( \ln \left( \frac{A_X}{A_L} \right) \) into an initial condition and the sum of previous growth rates:

\[
\ln \left( \frac{A_X}{A_L} \right) \approx \ln \left( \frac{A_X^0}{A_L^0} \right) + \sum_{i=1}^{t} g_i(P_t).
\]

The true functional form of \( g(P_t) \) is unknown. I use a linear specification in my estimation, \( g_t = \gamma + \theta P_t. \) This implies:

\[
\ln \left( \frac{A_X}{A_L} \right) \approx \ln \left( \frac{A_X^0}{A_L^0} \right) + \gamma t + \theta \sum_{i=1}^{t} (P_t).
\]

Plugging back into the regression equation we have:

\[
\ln \left( \frac{r_t}{w_t} \right) = \alpha - \frac{1}{\sigma} \gamma t - \frac{1}{\sigma} \theta \sum_{i=1}^{t} (P_i) + \frac{1}{\sigma} \ln (P_t), \quad (9)
\]

where \( \alpha = \ln \left( \frac{A_X^0}{A_L^0} \right) - \frac{1}{\sigma} \ln (\bar{X}). \) These assumptions lead me to the following regression equation:

\[
\ln \left( \frac{r_t}{w_t} \right) = \alpha + \beta_1 t + \beta_2 \sum_{i=1}^{t} (P_i) + \beta_3 \ln (P_t), \quad (10)
\]

where we can recover the parameters of interest \( \gamma, \theta, \) and \( \sigma \) from the \( \beta \)s. Specifically, \( \gamma = -\frac{\beta_1}{\beta_3}, \theta = -\frac{\beta_2}{\beta_3} \) and \( \sigma = \frac{1}{\beta_3}. \)

In a previous section, I alluded to the fact that the time trend in the regressions in Table 1 had meaning. In a world with no induced innovation \( (\theta = 0) \), the time trend will allow us to recover the average difference between land-specific and labor-specific innovation over the sample period. The term \( \sum_{i=1}^{t} (P_i) \), or the sum of past levels of population, allows us to determine the amount of induced innovation. This is easier to see if I rewrite equation (10) in differences:

\[
\Delta \ln \left( \frac{r_t}{w_t} \right) = \beta_1 + \beta_2 P_t + \beta_3 \Delta \ln (P_t), \quad (11)
\]

\footnote{Note that \( g_t \approx g_X - g_L. \)}