Dynamic Contracts with Random Monitoring

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Abstract

In environments where a principal contracts with many agents who each execute numerous independent tasks, it is often infeasible to evaluate an agent's performance on all tasks. Incentives under moral hazard are instead provided by monitoring only a subset of randomly selected tasks. We characterize optimal dynamic contracts implemented with this type of random monitoring technology. We consider a stochastic environment where the agent's cost of effort varies over time, and analyze situations where this cost is public or private information. In an optimal contract, the terms the agent is promised when monitoring reveals compliance are as good as when no monitoring is performed, and for some cost types are better. These latter types receive a monitoring reward. We also elicit the dynamics of contract parameters over time. As time passes and the agent becomes richer, the monitoring reward decreases as the threat of forgoing the promised stream of future compensation provides sufficient incentives for compliance.

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1 Introduction

Existing literature on dynamic contracting under moral hazard examines situations where in every period of the game, the principal observes a signal correlated with the action chosen by the agent in that period.[1] In certain contractual environments, particularly those where the principal interacts with a large number of agents who each performs numerous independent tasks over the lifetime of the contract, it is infeasible or too costly to make observations about an agent's performance on *all* tasks that he is assigned. Instead, in many such situations, the agent is provided with incentives by evaluating his performance only on a randomly generated subset of these tasks. In this paper we study optimal dynamic contracting under moral hazard when contracts are implemented with this type of *random monitoring* technology. Specifically, we consider an infinitely repeated game in which in every period the agent chooses a payoff relevant action (an effort level) and the principal employs a monitoring instrument which, with some non-degenerate probability, reveals the precise action chosen by the agent, but otherwise reveals no informative signal of it.

Optimal contracts with random monitoring are investigated in a *static* framework in Barbos (2016). An implicit assumption underlying static modeling of contractual environments is that a contract is signed and executed for every action that the agent performs. However, in many real-world situations where random monitoring is employed, the sort of physical constraints or cost structures that preclude monitoring the performance of an agent on all tasks are also likely to preclude *contracting* of each task. Instead, contracts are usually signed at the outset of a long-term relationship (for instance, at the time when a worker is hired) and then the agent is assigned a series of tasks over the period of the contract. In these situations, the principal can design contracts that incentivize the agent to expend resources on a task by means of both current period transfers and promises about future compensation. The goal of this paper is to derive properties of optimal contracts implemented with random monitoring in such dynamic environments.

[1] This signal may take the form of a noisy observation of the agent's action, but may also consist of an aggregate or average level of the agent's actions.

The sequential nature of interaction specific to typical real-world dynamic contractual relationships makes the assignment of tasks be done with a delay after the contract is signed. Therefore, the agent often learns the precise cost of executing a task only at the time when the task is assigned. We capture this feature in our model by assuming that the marginal cost of the resources (effort, time, etc.) spent by the agent in executing the current period action depends on a state of nature, referred to as his cost type, which evolves stochastically over time. The realizations of the cost type in different periods are independent and identically distributed. In the main specification of the model we consider the case where the type is publicly observed in every period.

An example of a situation captured by our model is that of a service provider whose workers execute a large number of tasks.[2] Such a worker, when dealing with a customer, chooses the level of customer satisfaction to deliver, interpreted as his action, while the variable complexity of the issue being addressed determines the cost of delivering that level. Monitoring is performed by the employer through feedback obtained from a random subset of customers served by the worker.[3]

In the recursive formulation of contracts that we employ in our analysis, the agent starts every period with a continuation value that captures the present value of the stream of future expected utilities promised to him. Accounting for this value, the contract specifies a recommended action for each possible realization of the agent's cost type, and a current period transfer and next period continuation value for each contingency that may emerge during the period of play.

A salient feature of contract design under random monitoring is that incentives to exert effort can only be provided with the contract variables specified for contingencies where monitoring is executed. These variables must thus be set sufficiently high. On the other hand, when the agent is risk averse, efficiency requires minimizing the variance in the sets of current period transfers and next period continuation values. These two goals are missalligned and therefore a key aspect of the analysis of contracts with random monitoring consists in eliciting the solution to this trade-off. A

[2] This example is borrowed from Barbos (2016).

[3] For instance, most car dealerships request feedback from the customers of their service center by asking them to complete a survey about general satisfaction and about more specific aspects, such as the quality of service performed by the manager, technician, billing department, etc. The imperfect response rate makes such monitoring random.

second trade-off, specific to problems of dynamic contracting, emerges from the option to deliver utility to the agent through both current and future transfers. The shape of the optimal contracts in this environment is determined by the solution to these two trade-offs and their interplay. Specifically, the first trade-off is solved by promising the agent a *monitoring reward* that comes in the form of a better contractual terms to a complying agent when monitoring is performed than when it is not. The second trade-off implies that this reward is decreasing in time as the agent gets richer.

Efficiency dictates that the current period transfers and the next period continuation values be constant across contingencies where they do not play a role in implementing the current period action. These contingencies are those where the recommended effort level is zero or where monitoring is not performed. Additionally, to minimize the intertemporal variance in the agent's income, the corresponding next period continuation values are set equal to the current period continuation value. The contract parameters specified for contingencies where a positive effort level is recommended and monitoring reveals compliance are set at the same common levels whenever these levels are sufficient to provide the agent with incentives to exert effort. For the remaining contingencies, incentive provision requires that these contract parameters be set at higher levels, while efficiency that the corresponding incentive constraints bind. Complying agents of certain cost types are thus promised a monitoring reward that comes in the form of a higher next period continuation value and current period transfer than the corresponding contract parameters with no monitoring.

A consequence of the optimal choice of next period continuation value is that as the agent's type fluctuates over time, this value stays constant at the end of periods with no monitoring and of periods where the incentive constraint for exerting effort does not bind, and increases at the end of the remaining periods.[4] The current period transfers are set equal to zero when the continuation value is small since the marginal cost of delivering utility to the agent through promises of future transfers is lower. When positive, the current period transfers increase as the continuation value increases over time. The monitoring reward is thus offered to complying agents at the beginning

[4] Unlike dynamic contracting models that assume standard noisy monitoring, where low realizations of the informative signal may decrease the continuation value in the next period, in our model, this value is non-decreasing.

of the contractual relationship when the low levels of the continuation value and current period transfer imply that the incentive constraint for effort provision binds. As the continuation value increases over time and the contract terms for contingencies where incentive provision is unnecessary improve, the size of this reward decreases and at some point is eliminated.

We also examine the dynamics of the effort level implemented in an optimal contract. At the beginning of the contractual relationship, when the incentive constraint for effort binds, this effort level is independent of the continuation value and is a function solely on the agent's efficiency. As the continuation value increases over time, the marginal cost of compensating the agent for effort increases. At some point, the effort level required from each type starts decreasing between periods in which the continuation value increases and fewer types are required to exert effort. When this value is sufficiently high, compensating the agent for effort under any state of the world becomes too costly. At that time, the agent is retired and thereafter made a constant transfer each period which maintains the continuation value at the same level for perpetuity.[5]

As an extension, we also study a model where the state of the world that determines the agent's random cost type is private information to the agent. As in the static model from Barbos (2016), we consider the case where the agent can communicate this state through a non-verifiable message and thus the principal can adjust current period transfers and next period continuation values as a function of this message. Our analysis elicits several qualitatitive features of the optimal contracts in this framework which are distinct from those under pure moral hazard.

First, under adverse selection the monitoring reward decreases in the cost efficiency of the agent, with the less efficient types being promised the same terms with monitoring as without when complying. Second, adverse selection induces inefficiency in the choice of implemented effort level. This level maximizes each type's virtual surplus that accounts for the information rents the agent has to be paid for his private type information, and is lower than the effort level from a situation with pure moral hazard that instead maximizes an intuitive notion of social surplus. Finally, when

[5] In models with noisy monitoring where the continuation value may decrease, retirement also occurs when this value reaches a lower bound (for instance, Spear and Wang (2005), Sannikov (2008)).

the agent is risk-neutral, unlike a situation with pure moral hazard where the first-best outcome can always be implemented, under adverse selection, this is possible only under an additional condition which ensures that the limited liability constraint of the agent is not violated. These findings show that some key qualitative features of the optimal contracts uncovered in the static version of the model with moral hazard and adverse selection extend to a dynamic environment.

This paper belongs to the vast literature of dynamic contracting under moral hazard. Some of the early seminal papers from this literature are Radner (1981), who examined a situation where the publicly observed period outcome depends on the agent's action and some state of the world which are both privately observed by the agent, Townsend (1982), which studied dynamic contracts under adverse selection, Rubinstein and Yaari (1983), which elicited the role of temporal incentives in mitigating the inefficiencies induced by moral hazard in repeated relationships, and Rogerson (1985), which showed that in a repeated moral hazard model where the agent can neither borrow nor save, the inverse marginal utility of consumption evolves as a martingale. Spear and Srivastava (1987) were the first to employ recursive methods for solving dynamic contracts. Some of the more recent contributions to this literature are Sannikov (2008), which studies contracting under moral hazard in a continuous time model, Biais, Mariotti, Rochet and Villeneuve (2010), which considers a model where the agent exerts effort to preempt large losses and his limited liability hinders risk sharing, and Kovrijnykh (2013) which studies the case where the principal has a limited ability to commit to enforce a contract. Our paper makes a contribution to this literature by considering situations where contracts are implemented with a random monitoring technology.[6] While some of the insights that it uncovers are familiar from earlier papers, other findings are specific due to the distinct set of contingencies on which contracts with random monitoring can be designed.

Section 2 introduces the model and some preliminaries, including the analysis of the full-information benchmark. Section 3 presents the analysis of the pure moral hazard model, while section 4 presents the analysis of the model with moral hazard and adverse selection. Section 5

[6] Fraser (2012) considers a two-period contracting problem under random monitoring in the context of agri-environmental policy. He analyzes numerically the effect of various model parameters on the incentives of the agent to deviate from the prescribed policy and illustrates the potential use by the principal of inter-temporal incentives.

concludes. Most proofs are relegated to the appendix.

2 Framework and Preliminaries

The Model There are two risk-neutral players, a principal ($\mathcal{P}$) and an agent ($\mathcal{A}$), who interact repeatedly over an infinite number of time periods and evaluate future streams of utility with a common discount factor $\beta \in (0,1)$. $\mathcal{P}$ owns a firm and can offer an employment contract to $\mathcal{A}$, which $\mathcal{A}$ may accept or reject. When $\mathcal{A}$ exerts effort e in a given period, he produces output with value $y(e)$, which is entirely appropriated by $\mathcal{P}$ through his firm. We assume $y(0) = 0$, $y'(\cdot) > 0$, $y''(\cdot) \leq 0$, and that and y is bounded on $\mathbb{R}_+$. If $\mathcal{A}$ receives a wage π and exerts effort e in a given period, his current period utility is $u(\pi) - c(s,e)$. The function $u : \mathbb{R} \to \mathbb{R}$ captures $\mathcal{A}$'s preferences over monetary transfers. We denote the corresponding inverse utility function by $h(u)$ and throughout the analysis substitute the current period induced utilities $u \equiv u(\pi)$ as choice variables in the optimal contract problem in place of wages. To distinguish u from the total utility experienced by $\mathcal{A}$ in a period, which also accounts for the cost of effort, we will refer in the following to u as the *wage-utility*. We assume $h'(\cdot) > 0$ and $h''(\cdot) \geq 0$, and normalize $h(0) = 0$. $c(s,e)$ is the cost for $\mathcal{A}$ of exerting effort e, where (i) s is a random variable, (ii) $c(\cdot,\cdot)$ is a function with $c(s,0) = 0$, $c_e > 0$, $c_{ee} > 0$, $c_s > 0$ and $c_{es} > 0$, for all $s \in [\underline{s}, \overline{s}]$ and $e \geq 0$. The realizations of the random variable s are independent over time and distributed with density function $f(\cdot)$ on an interval $[\underline{s}, \overline{s}] \subset \mathbb{R}$. The functions y, h and c are twice continuously differentiable.

We examine models with two specifications of the information structure. In a pure moral hazard model, the realization of the state s is public information in every period. As an extension, we also analyze a model with moral hazard and adverse selection in which the state s is observed only by $\mathcal{A}$; $\mathcal{A}$ can *non-verifiably* communicate the state and the corresponding message is contractible.

$\mathcal{P}$ has at his disposal an imprecise monitoring technology which allows publicly observing the effort exerted by $\mathcal{A}$ in a given period with some probability $r \in (0,1)$. With probability $1-r$, $\mathcal{P}$ does not observe either e or any informative signal about it. $\mathcal{P}$ incurs no additional cost when he observes the effort exerted by $\mathcal{A}$. $\mathcal{A}$ does not know at the time when he chooses the level of effort to exert in a given period whether or not $\mathcal{P}$ will observe this level.

Once $\mathcal{A}$ accepts a contract offered by $\mathcal{P}$, this contract is binding for the two players. $\mathcal{A}$'s outside option at the beginning of the employment relationship is normalized to have an expected present value equal to zero. We assume limited liability for the agent meaning that the wage that he receives in any period and under any contingency must be non-negative.

Recursive Approach As in other papers, we analyze this game by looking for the set of Pareto optimal subgame perfect equilibria.[7] Finding the Pareto equilibria requires solving an optimization problem where the expected equilibrium payoff of one player is maximized subject to delivering at least a certain expected equilibrium payoff to the other player. The standard approach to solving this problem in the context of dynamic games, introduced by Spear and Srivastava (1987), is to rewrite it in a recursive form with the current promised expected discounted value of future payoffs to one player as the state variable, and the contingent continuation values to that player, as well as the period actions by both players, as the control variables.

Contracts and Timing Since the discussion of how to transform a sequential optimization problem into its recursive representation is standard in the literature, we forgo including it here and instead assume directly that $\mathcal{P}$ offers *recursive contracts.* These contracts specify for any continuation value w at the beginning of a period and for each state $s \in [\underline{s}, \overline{s}]$, publicly observed or communicated by $\mathcal{A}$, a set $\{e_s, u_s(e), u_s^n, w_s(e), w_s^n\}$, where *(i)* e_s is the level of effort recommended to be exerted in the current period; *(ii)* $u_s(e)$ is the wage-utility delivered to $\mathcal{A}$ by the current period transfer promised for the case when monitoring is performed and the observed effort level is e; *(iii)* u_s^n is the wage-utility delivered to $\mathcal{A}$ by the current period transfer when monitoring is not performed; *(iv)* $w_s(e)$ is $\mathcal{A}$'s continuation value at the beginning of the next period if monitoring

[7] As a reminder, this is the set of subgame perfect equilibria with expected discounted payoffs for the two players such that there exist no other subgame perfect equilibrium in which both players enjoy weakly higher payoffs and at least one player a strictly higher payoff.

is performed and the observed effort is e; *(v)* w_s^n is $\mathcal{A}$'s continuation value at the beginning of the next period when monitoring is not performed. As standard in the literature, we allow $\mathcal{P}$ to offer probability mixtures over continuation values in any given contingency.[8]

We make now two straightforward observations that allow us to reduce the set of contracts that we consider; the formal arguments supporting these observations in a static framework from Barbos (2016) extend immediately to this dynamic setting. Thus, note first that in any optimal contract, it must be that $u_s(e) = 0$ and $w_s(e) = 0$ for all $e \neq e_s$, as this offers $\mathcal{A}$ the strongest incentives to choose effort level e_s. Since effort is costly, $\mathcal{A}$ will then either exert effort e_s or 0. Employing these remarks, in the following, we denote by u_s and w_s the current period wage-utility and continuation value, respectively, when monitoring reveals that $\mathcal{A}$ exerted the *recommended* effort level e_s.

With these observations, the timing of the contract in any period can be summarized as follows.

(i) At the beginning of the period, $\mathcal{A}$ is endowed with a continuation value w that may be the outcome of a lottery; in the first period, w equals 0. $\mathcal{P}$ presents a set of contract variables $\{e_s, u_s, u_s^n, w_s, w_s^n\}_{s\in[\underline{s},\bar{s}]}$, with an expected value to $\mathcal{A}$ of at least w when $\mathcal{A}$ complies with the effort recommendation for the state of nature to be realized in that period.

(ii) The random state s is realized. In the pure moral hazard model, s is publicly observed. In the model with moral hazard and adverse selection, s is observed only by $\mathcal{A}$. In the latter case, $\mathcal{A}$ non-verifiably communicates it to $\mathcal{P}$.

(iii) $\mathcal{A}$ exerts effort. Simultaneously, nature determines if monitoring is performed or not.

(iv) If monitoring is performed and the effort observed equals e_s, $\mathcal{A}$ is delivered a wage-utility u_s in the current period and is promised a continuation value for the next period w_s; if the effort is different than e_s the wage-utility and continuation value are set at 0. If monitoring is not performed, the wage-utility is u_s^n and the continuation value is w_s^n.

[8] This will imply that the value function in the recursive optimization problem is concave. Note that it is assumed that $\mathcal{P}$ can offer lotteries over continuation values in the next period (i.e., over future streams of utility) and not over current period wage-utilities. In fact, $\mathcal{P}$ has no reason to offer the latter, as it would require paying a risk premium.

The Full-Information Benchmark The first-best outcome in this framework corresponds to a situation with full information. If $\mathcal{P}$ observes both $\mathcal{A}$'s type and the effort exerted, monitoring plays no role. $\mathcal{P}$ thus offers a contract $\left\{e_s^0(t), u_s^0(t)\right\}_{s\in[\underline{s},\overline{s}]}$, where $e_s^0(t)$ is the effort required from type s in period t, and $u_s^0(t)$ is the wage-utility delivered to type s at the end of period t.

The next proposition presents the intuitive conditions defining the corresponding optimal contract when $\mathcal{A}$ is strictly risk averse, i.e., when $h'' > 0$. Since its proof employs concepts and results that are introduced in more generality in the proofs of the results from the model with unobserved effort, the proof is included in appendix A1 after the proofs of the results from section 3.

Proposition 1 *Assume $\mathcal{A}$ is strictly risk averse over monetary transfers. The full information optimal contract is stationary over time with $u_s^0(t) = u^0 > 0$ and $e_s^0(t) = e_s^0$ for all $s \in [\underline{s}, \overline{s}]$, where*

$$y'\left(e_s^0\right) - h'(u^0)c_e(s, e_s^0) \leq 0, \text{ and } = 0 \text{ if } e_s^0 > 0 \tag{1}$$

It is straightforward to show that if the agent is risk neutral, then in the optimal contract, the effort profile satisfies $c_e(s, e_s^0) = y'\left(e_s^0\right)$ for all $s \in [\underline{s}, \overline{s}]$ with $e_s^0 > 0$, while any nonnegative wage-utility profile $\left\{u_s^0\right\}_{s\in[\underline{s},\overline{s}]}$ satisfying $\int_{\underline{s}}^{\overline{s}} u_s^0 f(s)ds = \int_{\underline{s}}^{\overline{s}} c_e(s, e_s^0)f(s)ds$ is optimal.

3 The Pure Moral Hazard Model

Clearly, an optimal contract which induces strictly positive effort for some type s in a given period, must also induce strictly positive effort for all types that are more efficient than s.[9] Thus, for any continuation value w, there exists some value $\widehat{s} \in [\underline{s}, \overline{s}]$, such that, optimally, $e_s > 0$ a.e. $s \in [\underline{s}, \widehat{s})$,

[9]If types are distributed uniformly on $[\underline{s}, \overline{s}]$, a contract that does not satisfy this property can be improved by shifting required effort from a less efficient type to the more efficient one with zero effort while also swapping the remaining contract variables between the two types. All incentive constraints will be satisfied by the new contract, but $\mathcal{A}$ will experience a higher ex-ante utility since the cost of exerting the effort is lower. $\mathcal{P}$ can then appropriate at least partially this surplus by reducing the wages of the efficient types who are now working and whose incentive constraints are loose. The same type of adjustment can be performed when types are not uniformly distributed, but in that case, one needs to shift effort between two sets of types of equal measure.

and $e_s = 0$ a.e. $s \in [\widehat{s}, \overline{s}]$. Employing this observation, it follows that the recursive form of the optimal contract problem is

$$V(w) = \max_{\widehat{s}, \{e_s, w_s, w_s^n, u_s, u_s^n\}_{s \in [\underline{s}, \overline{s}]}} \int_{\underline{s}}^{\overline{s}} \{y(e_s) + r[\beta V(w_s) - h(u_s)] + (1-r)[\beta V(w_s^n) - h(u_s^n)]\} f(s)ds \tag{2}$$

$$\text{subject to: } r(\beta w_s + u_s) - c(s, e_s) \geq 0, \text{ for all } s \in [\underline{s}, \widehat{s}) \tag{ICE}$$

$$\int_{\underline{s}}^{\overline{s}} [r(\beta w_s + u_s) + (1-r)(\beta w_s^n + u_s^n) - c(s, e_s)] f(s)ds \ \geq w \tag{PKC}$$

$$e_s \geq 0,\ w_s \geq 0,\ w_s^n \geq 0,\ u_s \geq 0,\ u_s^n \geq 0, \text{ for all } s \in [\underline{s}, \overline{s}];\ e_s = 0, \text{ for all } s \in [\widehat{s}, \overline{s}] \tag{3}$$

Condition (ICE) (*Incentive Compatibility for Effort*) ensures that $\mathcal{A}$ exerts the recommended effort level in states $s \in [\underline{s}, \widehat{s})$. Condition (PKC) (*Promise-Keeping Constraint)* requires that the expected value of $\mathcal{A}$'s future stream of utilities, evaluated at the beginning of the current period, is at least w when accounting for the probability distribution of the current period wage-utility, effort, and next period continuation value. The constraints in (3) are consequences of $\mathcal{A}$'s limited liability and of the definition of $\widehat{s}$. The existence of a solution to problem (2)-(3) is studied in appendix A2.

To reduce the complexity of the notation in the text of the above problem and in the ensuing analysis, we forgo specifying explicitly throughout that the various payoffs may be random because $\mathcal{P}$ may offer lotteries over continuation values. Instead, with a slight abuse of notation, we write the expected value of a lottery over continuation values in place of the corresponding lottery.[10]

We examine first the case where $\mathcal{A}$ is strictly risk averse over monetary transfers, implying that h is strictly convex. The next lemma elicits some key properties of the value function V in any solution to (2)-(3). The proofs of all results from this section are presented in appendix A1.

Lemma 2 (Value Function) *The value function V is strictly decreasing and concave in w.*

[10] More precisely, this is the case when writing $\mathcal{A}$'s payoffs. For $\mathcal{P}$'s payoffs, when $\mathcal{P}$ offers a lottery $(w', p; w'', 1-p)$, with expected value w, we *can* write $V(w)$ instead of $pV(w') + (1-p)V(w'')$ because the two values are equal as V is *linear* on (w', w''). This is because $\mathcal{P}$ offers lotteries precisely when otherwise V would be convex on (w', w'') and thus $\mathcal{P}$ increases his expected payoff with a lottery over (w', w'') that linearizes the value function on that interval.

The fact that V is strictly decreasing in w is essentially equivalent to (PKC) always binding at the optimal solution. To understand the intuition for this fact, note that if (PKC) does not bind then (ICE) must bind for all types $s \in [\underline{s}, \widehat{s})$ since otherwise e_s could be increased so as to strictly increase the value of the objective function. Employing this insight, (PKC) can be rewritten as $(1-r)\int_{\underline{s}}^{\overline{s}} (\beta w_s^n + u_s^n) f(s) ds > w$. It follows then that the contract delivers $\mathcal{A}$ more than the promised utility w even though all surplus is delivered through the contract variables w_s^n and u_s^n, which do not play a role in implementing effort but only in minimizing the risk premium that $\mathcal{P}$ needs to pay. This cannot be optimal because slightly reducing these variables would increase the value of the contract. (PKC) must thus bind and therefore V is strictly decreasing in w. The proof of the lemma accounts for the fact that we do not know apriori that V is strictly decreasing, so the above argument may not necessarily lead to a contradiction if $u_s^n = 0$, but $w_s^n > 0$. Therefore, the argument in the appendix proceeds by showing first that V is strictly decreasing in w, from which it follows that (PKC) must bind. On the other hand, the concavity of the value function is a consequence of the fact that $\mathcal{P}$ can offer lotteries over the continuation values.

The properties of the optimal solution that are identified below hold on all $[\underline{s}, \overline{s}]$, but a set of measure zero. To avoid specifying this each time, in the following we restrict attention to the solution $\left(\widehat{s}, \{e_s, w_s, w_s^n, u_s, u_s^n\}_{s\in[\underline{s},\overline{s}]}\right)$ to (2)-(3) in which the properties we identify hold everywhere.

We will characterize the optimal contract in a series of propositions. To this goal, denote first by $V'_-(w)$, $V'_+(w)$ and $\partial V(w)$ the left derivative, the right derivative, and the superdifferential of the function V at w. Let also $\overline{w} \equiv \inf\left\{w \,\middle|\, V'_+(w) = -h'(0)\right\}$.

Proposition 3 (Optimal Continuation Values) *The optimal choice of continuation values satisfies: (i) $w_s^n = w$ for all $s \in [\underline{s}, \overline{s}]$; (ii) $w_s \geq w$ for all $s \in [\underline{s}, \overline{s}]$; (iii) $w_s = w$ for all $s \in [\underline{s}, \widehat{s})$ for which (ICE) does not bind, and for all $s \in [\widehat{s}, \overline{s}]$.*

Since V is concave, it is optimal to set the continuation value in the next period as close to w as possible. In particular, this continuation value is set at w for all contingencies where it plays no role in the incentive scheme that implements the current period effort, i.e., for contingencies where no monitoring is performed (thus, $w_s^n = w$ for all s) or where the recommended effort is 0 (thus, $w_s = w$ for $s \in [\widehat{s}, \overline{s}]$). On the other hand, the choice of next period continuation values for contingencies where positive effort is implemented and monitoring is performed is the solution to the trade-off between efficiency and incentive provision. When the incentive constraint (ICE) does not bind if w_s were set at w, this trade-off can be solved with no loss of efficiency. Otherwise, w_s must be set above w to provide sufficient incentives for the agent to follow the effort recommendation (as lemma 6 below elicits, for any s, this occurs for the lower values of w). To minimize the variance in the set of continuation values in such situations, w_s is set at the lowest level which satisfies the incentive constraint, and thus (ICE) binds. It is never optimal to set w_s below w as this induces an efficiency loss with no gain in the incentive scheme.

Proposition 4 (Optimal Wage-Utilities) *There exists $u^z \geq 0$ such that the optimal choice of current period wage-utilities satisfies: (i) $u_s^n = u^z$ for all $s \in [\underline{s}, \overline{s}]$; (ii) $u_s \geq u^z$ for all $s \in [\underline{s}, \overline{s}]$; (iii) $u_s = u^z$ for all $s \in [\underline{s}, \widehat{s})$ for which (ICE) does not bind, and for all $s \in [\widehat{s}, \overline{s}]$; (iv) $u^z = 0$ whenever $w \leq \overline{w}$; (v) for any $s \in [\underline{s}, \widehat{s})$, $u_s = 0$ whenever $w_s \leq \overline{w}$.*

By the same logic as that underlying proposition 3, since $\mathcal{A}$ is risk averse, the contract minimizes $\mathcal{A}$'s wage risk, and thus the current period wage-utilities are equal to some value u^z across all contingencies where they are not employed in incentive provision. In the remaining contingencies, the wage-utility is at least as high as u^z; otherwise, it could be increased while at the same time decreasing u^z to reduce the wage risk on $\mathcal{A}$ while still satisfying the two constraints. In a dynamic environment, at the beginning of every period, $\mathcal{P}$ can exploit the richness of the contract to provide incentives to $\mathcal{A}$ through both current period transfers and promises about future transfers. Parts *(iv)* and *(v)* of the proposition elicit a qualitative property of the optimal contract which is familiar from other dynamic principal-agent models.[11] Specifically, the current period wage-utilities are optimally set at 0 when the continuation value in the next period is low enough, i.e., given the

[11] See, for instance, Sannikov (2008) or Kovrijnykh (2013).

concavity of the value function V, when the marginal cost of providing incentives through promises of future transfers is low. The continuation value must reach $\overline{w}$ with probability one after any history of play since otherwise $\mathcal{A}$ never would consumes in that subgame; thus, $\overline{w}$ is finite.

Propositions 3 and 4 imply that if (ICE) binds for type s when the current continuation value is w, the expected lifetime utility delivered to a complying agent of type s when monitored is higher by an amount $(\beta w_s + u_s) - (\beta w + u^z) > 0$ than when not monitored. This difference constitutes a *monitoring reward* that the complying agent receives. As also elicited by Barbos (2016), this implies that when random monitoring is employed for incentive provision in a contractual relationship, it is optimal to not only to punish the agent for deviations from the prescribed effort level, but also to reward him when monitoring is executed and compliance is observed. This reward may come in the form of higher current period transfers and/or of promises about higher future compensation. Contracts that only punish observed deviations, but otherwise offer the same terms whether or not monitoring is executed, may not implement the desired level of effort in an optimal way.

Proposition 5 (Optimal Effort) *(i) The optimal choice of effort satisfies*

$$0 \in y'(e_s) + c_e(s, e_s)\, \partial V(w_s)\,,\ \textit{for } s \in [\underline{s}, \widehat{s}) \tag{4}$$

(ii) For any w, e_s is strictly decreasing in s for all $s \in [\underline{s}, \widehat{s})$.

Proposition 5 elicits properties of the optimal effort schedule. Condition (4) states the standard equality between the marginal benefit of implementing additional effort in some state s, $y'(e_s)$, and the marginal cost of compensating $\mathcal{A}$ for that effort, which is an element in the set $c_e(s, e_s)[-\partial V(w_s)]$, and when V is differentiable at w_s equals $-c_e(s, e_s) V'(w_s)$. As expected, $\mathcal{P}$ implements lower effort levels for less efficient types. As we argue in the appendix, the monotonicities of w_s and u_s with respect to s are generically ambiguous because, as s increases, the higher marginal cost of effort may be offset by the lower effort level that is recommended. The size of the monitoring reward is thus generically non-monotonic with respect to the efficiency of the agent.

Lemma 6 *For any s, there exists a value $\widetilde{w}_s$, such that (ICE) binds if and only if $w \leq \widetilde{w}_s$. Moreover, whenever $w \leq \widetilde{w}_s$, it is optimal to set $w_s = \widetilde{w}_s$.*

The threshold $\widetilde{w}_s$ is the minimum level at which the current period continuation value w provides sufficient incentives for effort to type s when setting the contract parameter w_s equal to w. Since the concavity of V requires the next period continuation value be as close to the current value as possible, $\widetilde{w}_s$ is precisely the level at which w_s is set whenever w is not sufficiently high and (ICE) binds.[12] An implication of lemma 6 is that $\mathcal{A}$ receives the monitoring reward for *low* values of w. When the agent is rich enough, the threat of possibly losing the value of the promised future compensation is sufficiently strong to induce compliance with the effort recommendation.

Proposition 7 describes the dynamics of the contract parameters over time.

Proposition 7 (Dynamics of the Contract Parameters) *The continuation value weakly increases over time. Whenever it increases between two periods to a new level w, the contract parameters have the following dynamics: (i) w_s stays constant at $\widetilde{w}_s$ if $w \leq \widetilde{w}_s$, and strictly increases if $w > \widetilde{w}_s$; (ii) w_s^n strictly increases; (iii) u_s stays constant if $w \leq \widetilde{w}_s$, and increases if $w > \widetilde{w}_s$; (iv) u_s^n increases; (v) e_s stays constant if $w \leq \widetilde{w}_s$, and decreases if $w > \widetilde{w}_s$; (vi) $\widehat{s}$ decreases.*

The dynamics of the current period continuation value are an immediate consequence of proposition 3 and lemma 6. By proposition 3, as the state fluctuates from period to period, the continuation value increases at the end of periods with monitoring and realizations of s such that $\widetilde{w}_s > w$, and stays constant at the end of all other periods. This value never decreases from period to period.

To understand parts *(i)*-*(iv)*, consider any pair of consecutive periods such that the continuation value at the beginning of the second period equals w with w strictly higher than the level of the continuation value at the beginning of the earlier of the two periods. By lemma 6, when w satisfies $w \leq \widetilde{w}_s$, it is optimal to set $w_s = \widetilde{w}_s$, implying that w_s stays constant between the two periods.

[12] Generically, $\widetilde{w}_s$ is non-monotonic in s and may take values below or above $\overline{w}$.

When $w > \widetilde{w}_s$, then $w_s = w$ and therefore w_s increases to w either from $\widetilde{w}_s$ or from the level of the continuation value in the first period. The fact that w_s^n increases between the two periods follows from the fact that w_s^n is always set equal to the current level of the continuation value. Since the marginal cost of implementing effort through promises about current and future transfers, i.e., though u_s and w_s, respectively, should optimally be equal whenever $u_s > 0$, the dynamics of u_s are identical to those of w_s. Similarly, the dynamics of u_s^n are identical to those of w_s^n when $u_s^n > 0$.

As part *(v)* states, for low values of w where constraint (ICE) binds, e_s is independent of w. To understand this, recall that the recommended effort level is determined by the trade-off between the marginal product and the marginal cost for $\mathcal{P}$ of implementing effort. By proposition 5*(i)*, the marginal cost is a function of the contract parameter w_s. Therefore, since whenever $w \leq \widetilde{w}_s$, w_s is set equal to $\widetilde{w}_s$, implying that w_s is independent of the current continuation value w, the effort level e_s is also independent of w. At the beginning of a contractual relationship, when the promised continuation value is low, $\mathcal{P}$ implements thus an effort level that is solely a function of the state s. As time passes and the continuation value satisfies $w > \widetilde{w}_s$ for some state s, the effort required from the agent of type $s \in [\underline{s}, \widehat{s})$ starts decreasing since the higher value of w increases the cost of compensating marginal effort. Moreover, by the same intuition, as w increases, $\mathcal{A}$ is required to exert effort under fewer realizations of the state variable s and thus $\widehat{s}$ decreases, as stated by *(vi)*.

An implication of the dynamics of these contract parameters is that the size of the monitoring reward promised to an agent of type s, $(\beta w_s + u_s) - (\beta w + u^z)$, decreases over time as the continuation value increases, and disappears when the continuation value reaches a level above $\widetilde{w}_s$.

The following proposition identifies the condition that defines the continuation value at which the agent is retired, meaning that he is no longer required to exert effort under any realization of the state. To this aim, denote by $\overline{\overline{w}} \equiv \inf \left\{ w \,\middle|\, V(w) = -\frac{1}{1-\beta} h((1-\beta)\, w) \right\}$.

Proposition 8 (Retirement Condition) *(i) For any w, $V(w) \geq -\frac{1}{1-\beta} h((1-\beta)\, w)$, with equality if $e_s = 0$ for all $s \in [\underline{s}, \overline{s}]$; (ii) For any $w \geq \overline{\overline{w}}$, $e_s = 0$ for all $s \in [\underline{s}, \overline{s}]$.*

When the continuation value reaches a value $w \geq \overline{\overline{w}}$, the agent is already rich enough that compensating him for effort becomes too costly. The agent is then retired and delivered utility equal to $(1-\beta)\,w$ every period, maintaining the continuation value at the beginning of every period constant at w. The resulting value of the contract to the principal is $-\frac{1}{1-\beta}h((1-\beta)\,w)$.[13] Spear and Wang (2005) and Sannikov (2008) identify a similar result in discrete-time and continuous-time, respectively, contracting problems under moral hazard. Since these papers assume a standard noisy monitoring technology, a sequence of low realizations of the output, as informative signals of the agent's effort, may lead the continuation value to reach a lower bound at which retirement is necessary since the incentivizing is not longer possible as the agent is too poor to be able to be punished if further low realizations of the output were to occur. Since in our model the continuation value is optimally non-decreasing, this second type of retirement condition does not emerge.

Proposition 9 (Monitoring Intensity) *$V(w)$ is increasing in r for all w and all r.*

The value of the contract is increasing in the intensity of monitoring. When r increases, $\mathcal{P}$ can rely less on the power of incentives and offer a smoother set of contract parameters. This reduces the dispersion in the set of possible continuation values and in the set of current period transfers, increasing the value of the contract since V is concave and $\mathcal{A}$ is risk averse. On the other hand, a higher value of r also increases the probability that $\mathcal{A}$ is paid the monitoring reward; this has an effect of reducing the value of the contract. Proposition 9 shows that the first effect dominates.

The closing result of the section states that the first-best value of a contract can be attained under moral hazard if $\mathcal{A}$ is risk neutral. The corresponding contract satisfies the incentive constraints solely through promises about the current period transfers for contingencies where monitoring is performed and $\mathcal{A}$ is required to exert positive effort. The remaining transfers as well as the continuation values are always set to 0.

[13] As shown in the appendix, the functions $V(w)$ and $-\frac{1}{1-\beta}h((1-\beta)\,w)$ also exhibit the so-called *smooth pasting property*, meaning that their first-order derivatives are equal at $\overline{\overline{w}}$ (V is shown to be differentiable at $\overline{\overline{w}}$).

Proposition 10 (Risk Neutral Agent) *If $\mathcal{A}$ is risk neutral, the value of the optimal contract under moral hazard equals the value of the optimal contract under full information.*

Proof. Let $w_s = 0$, $w_s^n = 0$ and $u_s^n = 0$ for all $s \in [\underline{s}, \overline{s}]$. Also, let $e_s = e_s^0$ and $u_s = \frac{1}{r}c(s, e_s^0)$ for all $s \in [\underline{s}, \overline{s}]$. If $\mathcal{A}$ is risk neutral, these contract parameters satisfy all the constraints in problem (2)-(3) for $w = 0$, therefore implementing the first-best outcome. Since $\mathcal{A}$'s outside option has value 0, $\mathcal{A}$ starts the first period of the game with $w = 0$, and therefore the first-best outcome can be implemented in that period. Moreover, by the specification of the continuation values, $\mathcal{A}$ will start the next period with $w = 0$. By induction, this contract implements the first-best outcome in all periods. Since the cost to $\mathcal{P}$ is equal to that under full information, this contract is optimal as its value attains the upper bound, i.e., the value of the optimal contract under full information. □

4 Extension: Model with Moral Hazard and Adverse Selection

As an extension of the pure moral hazard model analyzed in the previous section, we also examine a situation where the state of the world in any period of the game is observed privately by the agent and can be communicated to the principal through a non-verifyable message. The principal can offer contracts with parameters that are functions of this message. For simplicity, we assume in this section that it is optimal to implement strictly positive effort for all types. The additional insights uncovered by the analysis of a more general model that does not employ this assumption are similar to those derived in the static case examined in Barbos (2016).

In designing the optimal contract under moral hazard and adverse selection, $\mathcal{P}$ has to preempt two possible types of deviations by $\mathcal{A}$. First, $\mathcal{P}$ needs to ensure that an agent of type s does not communicate a message $\widetilde{s} \neq s$ and then exerts the effort recommended for $\widetilde{s}$. Second, $\mathcal{P}$ needs to ensure that $\mathcal{A}$ does not communicate some message $\widetilde{s}$ and then exerts no effort.[14] The recursive

[14] We employed again here our observation that $\mathcal{P}$ promises a wage equal to 0 for situations where monitoring reveals an effort level different than the one specified for the state of the world that $\mathcal{A}$ previously communicated.

form of the optimal contract problem is thus the following

$$V(w) = \max_{\{e_s, w_s, w_s^n, u_s, u_s^n\}_{s\in[\underline{s},\overline{s}]}} \int_{\underline{s}}^{\overline{s}} \{y(e_s) + r[\beta V(w_s) - h(u_s)] + (1-r)[\beta V(w_s^n) - h(u_s^n)]\} f(s)ds \tag{5}$$

$$\text{subject to: } s \in \arg\max_{\widetilde{s}\in[\underline{s},\overline{s}]} [r(\beta w_{\widetilde{s}} + u_{\widetilde{s}}) + (1-r)(\beta w_{\widetilde{s}}^n + u_{\widetilde{s}}^n) - c(s, e_{\widetilde{s}})], \text{ for all } s \in [\underline{s}, \overline{s}] \tag{ICT}$$

$$r(\beta w_s + u_s) + (1-r)(\beta w_s^n + u_s^n) - c(s, e_s) \geq (1-r) \max_{\widetilde{s}\in[\underline{s},\overline{s}]} (\beta w_{\widetilde{s}}^n + u_{\widetilde{s}}^n), \text{ for all } s \in [\underline{s}, \overline{s}] \tag{ICE}$$

$$\int_{\underline{s}}^{\overline{s}} [r(\beta w_s + u_s) + (1-r)(\beta w_s^n + u_s^n) - c(s, e_s)] f(s)ds \geq w \tag{PKC}$$

$$w_s \geq 0;\ w_s^n \geq 0;\ u_s \geq 0;\ u_s^n \geq 0 \tag{6}$$

Condition (ICT) (*Incentive Compatibility for Type*) preempts the first kind of deviation. Condition (ICE) (*Incentive Compatibility for Effort*) preempts the second kind; note that $(1-r)\max_{\widetilde{s}\in[\underline{s},\overline{s}]}(\beta w_{\widetilde{s}}^n + u_{\widetilde{s}}^n)$ is the highest expected payoff that $\mathcal{A}$ can obtain with an optimal choice of message and zero effort. The promise-keeping condition (PKC) has the same form as in the pure moral hazard model.

We examine first the case where $\mathcal{A}$ is risk averse over monetary transfers. The standard results from the literature of existence and uniqueness of a solution to recursive optimization problems do not apply to problem (5)-(6) because, under (ICT), the constraint set is not necessarily convex. Additionally, the optimal control tools we employ in our analysis require that the corresponding solution possess certain smoothness properties for which conditions on the primitives are difficult to identify. We perform our ensuing analysis under the assumption that there exists a solution $\{e_s, w_s, w_s^n, u_s, u_s^n\}_{s\in[\underline{s},\overline{s}]}$ to (5)-(6) which is differentiable in s, with the corresponding value function V continuously differentiable and strictly concave in w.[15] We denote in the following by e_s' the derivative of e_s with respect to s, and use a similar notation for the derivatives of other variables.

The following two results simplify problem (5)-(6). Lemma 11, whose proof is similar to that of the corresponding result in Barbos (2016), and thus omitted here, identifies a weaker sufficient

[15] The weak concavity of V would alternatively follow if $\mathcal{P}$ can offer probability mixtures over continuation values.

condition (ICEW) (*Weak Incentive Compatibility For Effort*) for a contract to preempt shirking. Specifically, it states that a contract which induces truthful type revelation, i.e., which satisfies (ICT), will induce all agent types to exert effort as long as it induces the least efficient type $\overline{s}$ to exert effort. Lemma 12 validates in this setting the standard First Order Approach to contracting problems with asymmetric information. Its proof is also similar to the proof of the result from Barbos (2016), and therefore omitted. Accounting for the result of lemma 12, as in the standard approach from the literature to such situations, in the following, we replace (ICT) with (ICTW) (*Weak Incentive Compatibility For Type*) and assume that the constraint in (7) is satisfied with strict inequality. Barbos (2016) presents the analysis of the case where the constraint may bind.

Lemma 11 *Any contract that satisfies (ICT) will satisfy (ICE) if and only if*

$$r\left(\beta w_{\overline{s}} + u_{\overline{s}}\right) + (1-r)\left(\beta w^n_{\overline{s}} + u^n_{\overline{s}}\right) - c\left(\overline{s}, e_{\overline{s}}\right) \geq (1-r)\left(\beta w^n_s + u^n_s\right), \textit{ for all } s \in [\underline{s}, \overline{s}] \qquad \text{(ICEW)}$$

Lemma 12 *A contract induces truthful type revelation for all $s \in [\underline{s}, \overline{s}]$ if and only if*

$$e'_s \leq 0 \ \textit{a.e.} \ \ s \in [\underline{s}, \overline{s}] \qquad (7)$$

$$r\left(\beta w'_s + u'_s\right) + (1-r)\left(\beta w^{n\prime}_s + u^{n\prime}_s\right) = c_e(s, e_s)e'_s \ \textit{a.e.} \ \ s \in [\underline{s}, \overline{s}] \qquad \text{(ICTW)}$$

As in section 3, denote by $\overline{w} \equiv \inf\{w | V'(w) = -h'(0)\}$. Proposition 13 derives features of the optimal contract which mirror properties of the contract under pure moral hazard, and have a similar intuition. The proofs of all results from this section are presented in appendix A3.

Proposition 13 *(i) For all $s \in [\underline{s}, \overline{s}]$, it is optimal to set $w_s \geq w^n_s$ and $u_s \geq u^n_s$; (ii) For any $s \in [\underline{s}, \overline{s}]$ for which (ICEW) does not bind, it is optimal to set $w_s = w^n_s$ and $u_s = u^n_s$; (iii) $u_s = 0$ if and only if $w_s < \overline{w}$; (iv) $u^n_s = 0$ if and only if $w^n_s < \overline{w}$; (v) $w_s > 0$ for all s;*

By parts *(i)* and *(ii)* of the proposition, a complying agent is always offered at least as high continuation values and wage-utilities when monitored as when not monitored; these variables are equal for the types for which the incentive constraint that induces effort does not bind. Parts *(iii)* and *(iv)* show that the trade-off between current and future payments is solved in the same manner as in a situation without adverse selection: the current period transfers are zero when the cost of delivering them through promises about future transfers is low. Part *(v)* is a consequence of the fact that w_s is the primary tool for incentivizing effort and therefore must be strictly positive since the contract implements strictly positive effort for all types.

Proposition 14 (Partition of Types) *There exists* $\overleftrightarrow{s} \in [\underline{s}, \overline{s}]$ *such that*

(i) for $s \in [\underline{s}, \overleftrightarrow{s})$*, we have* $w_s > w_s^n$*,* $u_s > u_s^n$ *if* $u_s > 0$*,* $w_s' < 0$*,* $u_s' \leq 0$*, and* $w_s^{n\prime} = u_s^{n\prime} = 0$*;*

(ii) for $s \in [\overleftrightarrow{s}, \overline{s}]$*, we have* $w_s = w_s^n$*,* $u_s = u_s^n$*,* $w_s' < 0$*, and* $u_s' \leq 0$*.*

By proposition 14, generically, the optimal contract partitions the set of types in two regions. The more efficient types in $[\underline{s}, \overleftrightarrow{s})$ are promised a monitoring reward and the monotonicities in s of the contract parameters elicited by part *(i)* imply that the net value of this reward, $(\beta w_s + u_s) - (\beta w_s^n + u_s^n)$, is strictly decreasing in s. The remaining, less efficient, types from $[\overleftrightarrow{s}, \overline{s}]$ are promised the same continuation value and wage-utility when monitoring reveals compliance as when not monitored. This partition of types is a feature of the optimal contract that also emerges in static version of this model from Barbos (2016) and is in both models a consequence of the private information about his cost type that the agent is endowed with. It is perhaps slightly counterintuitive a priori, as one may expect more efficient types to be easier to incentivize. However, as elicited by lemma 12, to ensure incentive compatibility under adverse selection, the effort required from these types must also higher and this effect dominates.

To understand the results of proposition 14, note first that (7) and (ICTW) imply that a contract that induces truthful revelation of $\mathcal{A}$'s private cost information must satisfy the property that $r\left(\beta w_s + u_s\right) + (1-r)\left(\beta w_s^n + u_s^n\right)$ is decreasing in s. If $w_s = w_s^n$ and $u_s = u_s^n$, this implies that $\beta w_s^n + u_s^n$ must be decreasing in s (and thus also $\beta w_s + u_s$). Therefore, once (ICEW) is satisfied for some s, and thus by proposition 13, $w_s = w_s^n$ and $u_s = u_s^n$, (ICEW) will also be satisfied for all higher values of s. The monotonicities of w_s^n and u_s^n on $[\underline{s}, \overleftrightarrow{s})$ follow from the fact that it is feasible to provide incentives for truthful type revelation by employing only w_s and u_s, and thus to minimize the variance in the set of future continuation values and $\mathcal{A}$'s risk exposure, respectively, w_s^n and u_s^n are optimally constant on $[\underline{s}, \overleftrightarrow{s})$. From these, it follows that $\beta w_s + u_s$ is decreasing in s on the whole interval $[\underline{s}, \overline{s}]$. The fact that both w_s and u_s are in fact decreasing in s is then a consequence of the fact that, optimally, the marginal costs of delivering utility to $\mathcal{A}$ through current and future transfers, respectively, must be equal whenever $u_s > 0$.

Proposition 15 (Optimal Effort) *For any $s \in [\underline{s}, \overline{s}]$, the optimal effort choice satisfies*

$$-V'(w_s)c_e(s, e_s)\, f(s) + c_{es}(s, e_s) \int_{\underline{s}}^{s} \left[V'(w) - V'(w_\sigma)\right] f(\sigma) d\sigma = y'(e_s)\, f(s) \tag{8}$$

Remark 16 *We have $\int_{\underline{s}}^{s} [V'(w) - V'(w_\sigma)]\, f(\sigma)d\sigma > 0$ for all $s \in (\underline{s}, \overline{s})$.*

Proposition 15 elicits the condition that defines the optimal effort choice as a function of the rest of the contract. As in (4), this condition equates the marginal cost and benefit of implementing additional effort, while accounting for the additional cost in terms of information rents that $\mathcal{A}$ is paid under adverse selection. As in the static model, moral hazard induces a departure from the efficient outcome by subjecting $\mathcal{A}$ to risk, while adverse selection induces inefficiency in the effort level. For the given profile of continuation values, e_s maximizes type s's virtual surplus, $y(e)f(s) - V'(w_s)c(s, e_s)\, f(s) - c_s(s, e) \int_{\underline{s}}^{s} [V'(w) - V'(w_\sigma)]\, f(\sigma)d\sigma$. Unlike a situation with pure moral hazard, $\mathcal{P}$ cannot implement the effort that maximizes the social surplus, $y(e) - V'(w_s)c(s, e_s)$. Instead, $\mathcal{P}$ implements a *lower*[16] level of effort under adverse selection.

Proposition 17 (Monitoring Intensity) *$V(w)$ is increasing in r for all w and all r.*

As with pure moral hazard, the value of the contract is increasing in the intensity of monitoring.

[16] This follows from $c_{es} > 0$, $c_{ee} > 0$, $y'' < 0$, V concave, and result of the remark 16.

The last result of the section identifies a condition under which the first best outcome can be implemented under moral hazard and adverse selection when the agent is risk neutral. This condition ensures that the limited liability constraint of the agent is not violated by a contract that induces truthful type revelation and implements the first best effort level. As in the rest of the analysis in this section, we assume that $e_s^0 > 0$ for all $s \in [\underline{s}, \overline{s}]$, i.e., that it is optimal to implement strictly positive effort in all states under full information.

Proposition 18 (Risk Neutral Agent) *If $\mathcal{A}$ is risk neutral and $c\left(\overline{s}, e_{\overline{s}}^0\right) - \int_{\underline{s}}^{\overline{s}} c_s\left(s, e_s^0\right) F(s)ds \geq 0$, the value of the optimal contract under moral hazard and adverse selection equals the value of the optimal contract under full information.*

5 Conclusion

In this paper we characterize optimal contracts with random monitoring in a repeated contractual relationship where incentives can be provided both with current transfers and promises about future transfers. The shape of the optimal contract is determined by the solution to two trade-offs that emerge in this context and their interplay. The first trade-off is between incentive provision and efficiency. Specifically, since incentives to exert effort under random monitoring can only be provided with the contract variables specified for contingencies where monitoring is executed, these variables must be sufficiently high, whereas efficiency requires minimizing the variance in the agent's current and future income. The second trade-off is that between delivering utility to the agent through current and future transfers. The solution to the first trade-off involves promising the agent a reward when monitoring reveals compliance relative to the terms of the contract promised when no monitoring is performed. This reward comes in the form of higher current period transfers of promises about future compensation. As time passes, the size of the reward decreases as the agent becomes rich enough that the threat of a loss of the promised stream of future income provides the agent with sufficient incentives for complying with the effort recommendation.

Appendix

Appendix A1. Proofs for Section 3.

Proof of Lemma 2 Clearly V is decreasing since a higher value of w tightens the constraint set. We argue next that V is in fact strictly decreasing. Assume by contradiction that this is not the case and that for some $w_1 \in [0, \infty)$, there exists $w > w_1$ with $V(w) = V(w_1)$. Let $w_2 \equiv \sup \{w \geq w_1 | V(w) = V(w_1)\}$.[17] Denote by $\mathbf{P}(w)$ the problem (2)-(3) for a given value of w. Let χ^1 and χ^2 be the maximizers of $\mathbf{P}(w_1)$ and $\mathbf{P}(w_2)$, respectively (χ is the set of variables $\widehat{s}, \{e_s, w_s, w_s^n, u_s, u_s^n\}_{s\in[\underline{s},\overline{s}]}$). Since χ^2 satisfies the constraints of $\mathbf{P}(w_1)$ and it generates the same value of the objective function as χ^1, it must be that χ^2 is a maximizer of $\mathbf{P}(w_1)$. Moreover, (PKC) does not bind when the solution to $\mathbf{P}(w_1)$ is χ^2, since χ^2 satisfies (PKC) for $w = w_2$. Next we will derive properties of χ^2 as determined by the fact that it is a solution to $\mathbf{P}(w_1)$ with (PKC) non-binding. First, (ICE) must bind for $s \in [\underline{s}, \widehat{s}]$ a.e., since otherwise e_s could be increased slightly, which since (PKC) is loose, would not violate (PKC). Second, it must be that $u_s^n = 0$ for $s \in [\underline{s}, \overline{s}]$ a.e., since otherwise u_s^n could be decreased. Employing these two observations, the non-binding (PKC) implies $(1-r)\,\beta \int_{\underline{s}}^{\overline{s}} w_s^n f(s)ds > w_1$. Moreover, since χ^2 satisfies (PKC) for $w = w_2$, we have $(1-r)\,\beta \int_{\underline{s}}^{\overline{s}} w_s^n f(s)ds \geq w_2$. Thus, for any optimal profile $\{w_s^n\}_{s\in[\underline{s},\overline{s}]}$, there must exist a set of strictly positive measure $\Phi \subset [\underline{s}, \overline{s}]$ such that $w_s^n \geq \frac{w_2}{(1-r)\beta} > w_2$ for all $s \in \Phi$, implying by the choice of w_2 that $V(w_s^n) < V(w_2) = V(w_1)$.

We will argue next that there is an adjustment of $\{w_s^n\}_{s\in[\underline{s},\overline{s}]}$ in the solution χ^2 to problem $\mathbf{P}(w_1)$ that would increase the value of the objective function while still satisfying the constraints. This will then constitute the desired contradiction. Consider thus the profile $\{\widehat{w}_s^n\}_{s\in[\underline{s},\overline{s}]}$, with $\widehat{w}_s^n \equiv w_s^n - \varepsilon$ for $s \in \Psi \subset \Phi$, and $\widehat{w}_s^n \equiv w_s^n$ for $s \in [\underline{s}, \overline{s}] \backslash \Psi$, where ε and Ψ are chosen so that $\{\widehat{w}_s^n\}_{s\in[\underline{s},\overline{s}]}$ satisfies the following three conditions. The first condition, $(1-r)\,\beta \int_{\underline{s}}^{\overline{s}} \widehat{w}_s^n f(s)ds \geq w_1$, will ensure that constraint (PKC) continues to be satisfied. The second condition is $\widehat{w}_s^n \leq w_2$ for

[17] Clearly, w_2 is finite since otherwise $V(w_2) = -\infty$, which implies that $V(w_1) = -\infty$, and thus that $w_1 = -\infty$.

all $s \in \Psi$; since this implies that $V(\widehat{w}_s^n) \geq V(w_2) > V(w_s^n)$, it will follow that this adjustment strictly increases the value of the objective function in problem $\mathbf{P}(w_1)$. The third condition is that $\widehat{w}_s^n \geq 0$, i.e., that $\varepsilon \leq w_s^n$. Now, since $(1-r)\,\beta \int_{\underline{s}}^{\overline{s}} w_s^n f(s)ds \geq w_2$, we have $(1-r)\,\beta \int_{\underline{s}}^{\overline{s}} \widehat{w}_s^n f(s)ds \geq w_2 - (1-r)\,\beta\varepsilon \int_\Psi f(s)ds$, and thus for the first condition to be satisfied, we need $\varepsilon \leq \frac{w_2 - w_1}{(1-r)\beta \int_\Psi f(s)ds}$ (*). The second condition can be rewritten as $w_s^n - w_2 \leq \varepsilon$. Since $w_s^n \geq \frac{w_2}{(1-r)\beta} > w_2$ for all $s \in \Phi$ implies $w_s^n - w_2 \geq \frac{w_2}{(1-r)\beta} - w_2$, for the second condition to be satisfied is would be enough to ensure that $\varepsilon \geq \frac{w_2}{(1-r)\beta} - w_2$ (**). Finally, to satisfy the third condition, since $w_s^n \geq \frac{w_2}{(1-r)\beta}$ for all $s \in \Phi$, it is enough that $\varepsilon \leq \frac{w_2}{(1-r)\beta}$ (***). Thus, a value ε exists satisfying conditions (*), (**), and (***) if $\min\left\{\frac{w_2 - w_1}{(1-r)\beta \int_\Psi f(s)ds}, \frac{w_2}{(1-r)\beta}\right\} > \frac{w_2}{(1-r)\beta} - w_2$, i.e., $w_2 - w_1 > w_2\left[1 - (1-r)\,\beta\right] \int_\Psi f(s)ds$. This can be ensured by choosing the set Ψ of sufficiently small, but strictly positive measure. This completes the proof of the fact that V is strictly decreasing.

Note that this immediately implies that (PKC) must bind for all w in a solution to problem (2)-(3) since otherwise V would be constant on intervals where (PKC) does not bind. □

We continue by introducing several preliminary concepts and results that will be employed in the ensuing analysis for comparative statics of the contract variables.

Definition 19 *A correspondence $H : \mathbb{R} \to 2^{\mathbb{R}}$ is increasing (strictly increasing) if for any $x > y$, and $x' \in H(x)$, $y' \in H(y)$, we have $x' \geq y'$ $(x' > y')$.*

Lemma 20 *Let $\overline{V}$ be a convex function with subdifferential $\partial\overline{V}$ and $g(x)$ be a increasing function. Then the correspondence $\partial\overline{V}(g(x))$ is increasing in x.*

Proof. This follows from the fact that $\partial\overline{V}(g(x)) = \left\{\tau \in \mathbb{R} \,\middle|\, \overline{V}'_-(g(x)) \leq \tau \leq \overline{V}'_+(g(x))\right\}$ and Theorem 24.1 in Rockafellar (1970) which states that if $z_1 < \tau < z_2$ and $\overline{V}$ is convex then $\overline{V}'_+(z_1) \leq \overline{V}'_-(\tau) \leq \overline{V}'_+(\tau) \leq \overline{V}'_+(z_2)$. In particular, note that if $x_2 > x_1$, and thus $g(x_2) > g(x_1)$, then the theorem implies $\overline{V}'_+(g(x_1)) < \overline{V}'_-(g(x_2))$, which given the definition of the subdifferential implies that for any $\tau_1 \in \partial\overline{V}(g(x_1))$and $\tau_2 \in \partial\overline{V}(g(x_2))$, it must be that $\tau_1 \leq \tau_2$. □

Lemma 21 *Let $H(s,e)$ be a correspondence which is increasing in s and strictly increasing in e. For any s, define the correspondence $e_s \equiv \{e\,|0 \in H(s,e)\}$. Then e_s is a decreasing function of s. If in addition, $H(s,e)$ is strictly increasing in s, then e_s is a strictly decreasing function.*

Proof. Consider two values $s_2 > s_1$ and let $\tau_1 \in e_{s_1}$ and $\tau_2 \in e_{s_2}$. If $\tau_2 > \tau_1$, then for any $x_2 \in H(s_2,\tau_2)$ and $x_1 \in H(s_1,\tau_1)$, we would have $x_2 > x_1$ given the monotonicity of H in the two variables. But then $0 \in H(s_2,\tau_2) \cap H(s_1,\tau_1)$, as required by the definition of e_s, could not be satisfied. We conclude then that it must be that $\tau_2 \leq \tau_1$. Moreover, if $H(s,e)$ is strictly increasing in s then $\tau_2 < \tau_1$. These imply that e_s is a decreasing correspondence in s, and strictly decreasing if $H(s,e)$ is strictly increasing in s. By a similar argument, since H is strictly increasing in e, it follows that for any s, the set $\{e\,|0 \in H(s,e)\}$ must be a singleton, and thus e_s is a function. □

To solve problem (2)-(3), we first rewrite (ICE) as $\mathbf{1}_{s<\widehat{s}}\left[r\left(\beta w_s + u_s\right) - c\left(s, e_s\right)\right] \geq 0$, for all $s \in [\underline{s}, \overline{s}]$, and let $\mu_s f(s)$ be the nonnegative Lagrange multipliers associated with it. Also, we let $\gamma \geq 0$ be the multiplier on (PKC). The Lagrangian associated with problem (2)-(3) is then

$$\begin{aligned}\mathcal{L} \;\equiv\; & \int_{\underline{s}}^{\overline{s}} \left\{y\left(e_s\right) + r\left[\beta V(w_s) - h\left(u_s\right)\right] + (1-r)\left[\beta V\left(w_s^n\right) - h\left(u_s^n\right)\right]\right\} f(s)ds \\ & + \int_{\underline{s}}^{\overline{s}} \mathbf{1}_{s<\widehat{s}}\mu_s\left[r\left(\beta w_s + u_s\right) - c\left(s, e_s\right)\right] f(s)ds + \\ & + \gamma\left\{\int_{\underline{s}}^{\overline{s}} \left[r\left(\beta w_s + u_s\right) + (1-r)\left(\beta w_s^n + u_s^n\right) - c\left(s, e_s\right)\right] f(s)ds \;-w\right\}\end{aligned}$$

The corresponding necessary first-order conditions (cancelling out $f(s)$, r, $1-r$, or β) are

$$\frac{\partial \mathcal{L}}{\partial u_s} = -h'(u_s) + \mathbf{1}_{s<\widehat{s}}\mu_s + \gamma \leq 0, \text{ and } = 0 \text{ if } u_s > 0 \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial u_s^n} = -h'(u_s^n) + \gamma \leq 0, \text{ and } = 0 \text{ if } u_s^n > 0 \tag{10}$$

$$\frac{\partial \mathcal{L}}{\partial w_s} = 0 \Rightarrow \mathbf{1}_{s<\widehat{s}}\mu_s + \gamma \in -\partial V(w_s), \text{ if } w_s > 0 \tag{11}$$

$$\frac{\partial \mathcal{L}}{\partial w_s^n} = 0 \Rightarrow \gamma \in -\partial V(w_s^n), \text{ if } w_s^n > 0 \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial e_s} = y'(e_s) - \mathbf{1}_{s<\widehat{s}}\mu_s c_e\left(s, e_s\right) - \gamma c_e\left(s, e_s\right) \leq 0, \text{ and } = 0 \text{ if } s \in [\underline{s}, \widehat{s}) \tag{13}$$

where for (13) we used the fact that $e_s > 0$ if and only if $s \in [\underline{s}, \widehat{s})$. As one cannot generically guarantee the differentiability of V,[18] the conditions (11) and (12) are written in terms of its superdifferential. In addition, the necessary first order conditions with respect to w_s and w_s^n, for the cases where $w_s = 0$ and $w_s^n = 0$, respectively, are $\frac{\partial \mathcal{L}}{\partial w_s} \leq 0 \Rightarrow \mathbf{1}_{s<\widehat{s}}\mu_s + \gamma + V'(w_s) \leq 0$, if $w_s = 0$, and $\frac{\partial \mathcal{L}}{\partial w_s^n} \leq 0 \Rightarrow \gamma + V'(w_s^n) \leq 0$, if $w_s^n = 0$. Finally, by the Envelope Theorem with respect to w, we have that $\frac{\partial \mathcal{L}}{\partial w} = -\gamma \in \partial V(w)$.

Proof of Proposition 3 Since from the Envelope Theorem we have $\gamma \in -\partial V(w)$, (12) implies that, for all $s \in [\underline{s}, \overline{s}]$, if $w_s^n > 0$, then it must be that $\partial V(w) \cap \partial V(w_s^n) \neq \varnothing$. Since V is concave, it is therefore optimal to set $w_s^n = w$ (if V is strictly concave in a neighbourhood around w, then setting $w_s^n = w$ is the *unique* optimal choice; if V is linear around w, then this is one of the optimal choices for w_s^n). On the other hand, if $w_s^n = 0$ then the necessary first order condition with respect to w_s is $\gamma + V'(w_s^n) \leq 0$, which since $w_s^n = 0$ implies that $V'(0) \leq -\gamma \in \partial V(w)$. Since V is concave and the superdifferential of a concave function is a decreasing correspondence (this is an immediate implication of Theorem 24.1 in Rockafellar (1970), stated in the proof of lemma 20 above), it follows that $w \leq 0$, i.e., that $w = 0$. We conclude that in both situations, it must be that $w_s^n = w$, for all $s \in [\underline{s}, \overline{s}]$. This demonstrates part *(i)* of the proposition. Similarly, for all $s \in [\widehat{s}, \overline{s}]$, from (11) and the corresponding first order condition for the case where $w_s = 0$, imply that $w_s = w$ for all $s \in [\widehat{s}, \overline{s}]$, demonstrating the statements from part *(ii)* of the proposition for these types. Now, for the remaining statements from parts *(ii)* and *(iii)* that pertain to $s \in [\underline{s}, \widehat{s})$, note that $\gamma \in -\partial V(w)$ and (11) imply for all $s \in [\underline{s}, \widehat{s})$ with $w_s > 0$ that

$$\mu_s \in -\partial V(w_s) + \partial V(w) \tag{14}$$

which since $\mu_s \geq 0$ and V is concave implies $w_s \geq w$. In addition, when (ICE) does not bind and thus $\mu_s = 0$, it is optimal to set $w_s = w$. Finally, note that if $w_s = 0$ for some $s \in [\underline{s}, \widehat{s})$, then the

[18] The standard results that deliver differentiability require that the solution be interior to the constrained set. Since this is not necessarily the case here, we cannot apply results such as Theorem 4.11 in Stokey and Lucas (1989) to conclude that V is differentiable.

necessary first order condition is $\mu_s + \gamma + V'(w_s) \le 0$. This implies that $\mu_s + V'(0) \le -\gamma \in \partial V(w)$, and thus since $\mu_s \ge 0$ and $\gamma \ge 0$ that $w = 0$ (since $V'(0) \le -\gamma \in \partial V(w)$ and V concave) and then that $\gamma = \mu_s = 0$. This completes the proof of the proposition 3. $\square$

Proof of Proposition 4 First, from (10) it follows that u_s^n is constant in s and thus part *(i)* of the proposition. To see this, note that the only case where this is not immediate is if $u_s^n > 0$ for $s \in S$, and $u_s^n = 0$ for the remaining $s \in [\underline{s}, \overline{s}] \backslash S$, where S is a proper subset of $[\underline{s}, \overline{s}]$. Now, if this was the case, then (10) would imply that $u_s^n = u^z > 0$ for $s \in S$, $\gamma = h'(u^z)$ and $\gamma \le h'(0)$. From the latter two implications, we would have $h'(u^z) \le h'(0)$, and thus $u^z = 0$ since h is convex; this delivers the contradiction. Parts *(ii)* and *(iii)* follow from (9) and (10), the fact that h is convex, and that $\mu_s \ge 0$ with $\mu_s = 0$ when (ICE) does not bind.

Now, for part *(iv)*, note that when $w < \overline{w}$, for any $-\gamma \in \partial V(w) \equiv \left\{\tau \in \mathbb{R} \,\middle|\, V'_+(w) \le \tau \le V'_-(w)\right\}$ and any $u \ge 0$, we have that $\gamma \le -V'_+(w) \le -V'_+(\overline{w}) = h'(0)$, where the second inequality follows from the fact that $-V$ is convex and from Theorem 24.1 in Rockafellar (1970), while the equality from the definition of $\overline{w}$. Therefore, $-h'(u) + \gamma \le -h'(u) + h'(0)$, which since h is strictly increasing, implies $-h'(u) + \gamma < 0$ for any $u > 0$. It follows then from (9) and (10) that it must be that $u_s = 0$ for any $s \in [\widehat{s}, \overline{s}]$, and that $u_s^n = 0$ for any $s \in [\underline{s}, \overline{s}]$, respectively.

Finally, for part *(v)* of the proposition, if $w_s < \overline{w}$ for some $s \in [\underline{s}, \widehat{s})$, we have for any $u > 0$ that $-h'(u) + \mu_s + \gamma < 0$ since by (11) $\mu_s + \gamma \in -\partial V(w_s)$, and thus $\mu_s + \gamma < -V'_+(\overline{w}) = h'(0)$. Therefore, if $w_s < \overline{w}$, then $u_s = 0$ as stated by part *(v)*. Note also here that if $w \le \overline{w}$ and $\mu_s = 0$, then $u_s = 0$, whereas if $\mu_s > 0$, then it is possible that $u_s > 0$ since the condition $-h'(0) + \mu_s + \gamma = 0$ is attained at smaller values of w than $\overline{w}$. We cannot, thus, substitute w for w_s in the text of part *(v)*. $\square$

Proof of Proposition 5 Part *(i)* of the proposition follows immediately from (11) and (13).

We will prove next part *(ii)*. Consider first any value $w \in [0, \overline{w})$ and any $s \in [\underline{s}, \widehat{s})$ with $u_s = 0$. When (ICE) does *not* bind for some type s, then as shown in proposition 3*(iii)*, we have $w_s = w$.

Substituting this into (4), it follows that e_s must satisfy

$$0 \in -y'(e_s) + c_e\left(s, e_s\right)\left[-\partial V(w)\right] \tag{15}$$

Since $\partial V(w)$ is independent of s and e, it follows that the correspondence $H(s,e) \equiv -y'(e) + c_e\left(s,e\right)\left[-\partial V(w)\right]$ is strictly increasing in e and s (for the latter, we use the fact that $\partial V(w) \subset (-\infty, 0)$). Therefore, by lemma 21 e_s is strictly decreasing in s. When (ICE) binds, by our assumption that $u_s = 0$, we have $w_s = \frac{1}{r\beta} c\left(s, e_s\right)$, and thus from (11) it follows $\mu_s + \gamma \in -\partial V\left(\frac{1}{r\beta} c\left(s, e_s\right)\right)$. Substituting $\mu_s + \gamma$ from (13) (which is satisfied with equality since $e_s > 0$ when $s \in [\underline{s}, \widehat{s})$) into this, we obtain that e_s satisfies $\frac{y'(e_s)}{c_e(s,e_s)} \in -\partial V\left(\frac{1}{r\beta} c\left(s, e_s\right)\right)$ which can be rewritten as

$$0 \in -y'(e_s) + c_e\left(s, e_s\right)\left[-\partial V\left(\frac{1}{r\beta} c\left(s, e_s\right)\right)\right] \tag{16}$$

Now, $c_e > 0$ and the fact that $\overline{V} \equiv -V$ is strictly increasing and convex imply that the correspondence $\partial \overline{V}\left(\frac{1}{r\beta} c\left(s, e\right)\right)$ is increasing in e and s by lemma 20. Let $H(s,e) \equiv -y'(e) + c_e\left(s,e\right)\left[-\partial V\left(\frac{1}{r\beta} c\left(s,e\right)\right)\right] = \left\{-y'(e) + c_e\left(s,e\right)x \,\middle|\, x \in -\partial V\left(\frac{1}{r\beta} c\left(s,e\right)\right)\right\}$, and note that $y'' < 0$, $c_{ee} > 0$, $c_s > 0$, $c_{es} > 0$ and $\partial V\left(\frac{1}{r\beta} c\left(s,e\right)\right) \subset (-\infty, 0)$ imply that $H(s,e)$ is *strictly* increasing in e and s. Therefore, again by lemma 21, e_s is strictly decreasing in s.

Consider now any value $w \geq \overline{w}$ or any value $w \in [0, \overline{w})$ and $s \in [\underline{s}, \widehat{s})$ such that $u_s > 0$, i.e., values of w for which transfers towards the agent under various contingencies are not *all* zero. If (ICE) does not bind for some type s, then the analysis of the comparative statics of e_s with respect to s is identical to that of the corresponding case for situations where $u_s = 0$ since these induced utilities do not enter the equation (15) that defines the optimal value of e_s. If (ICE) binds, then $w_s = \frac{1}{\beta r} c\left(s, e_s\right) - \frac{u_s}{\beta}$. From (9), since $u_s > 0$ we have $\mu_s + \gamma = h'(u_s)$, which substituted into (13) implies $y'(e_s) - h'(u_s) c_e\left(s, e_s\right) = 0$, and therefore that $u_s = (h')^{-1}\left(\frac{y'(e_s)}{c_e(s,e_s)}\right)$. On the other hand, since (9) and (11) imply that $h'(u_s) \in -\partial V(w_s)$, it follows by substituting terms derived above into $y'(e_s) - h'(u_s) c_e\left(s, e_s\right) = 0$ that $\frac{y'(e_s)}{c_e(s,e_s)} \in -\partial V\left(\frac{1}{\beta r} c\left(s, e_s\right) - \frac{1}{\beta}\left(h'\right)^{-1}\left(\frac{y'(e_s)}{c_e(s,e_s)}\right)\right)$, which can be rewritten as

$$0 \in -y'(e_s) + c_e(s,e_s)\left[-\partial V\left(\frac{1}{\beta r}c(s,e_s) - \frac{1}{\beta}(h')^{-1}\left(\frac{y'(e_s)}{c_e(s,e_s)}\right)\right)\right] \tag{17}$$

Note now that

$$\frac{\partial}{\partial e}\left[\frac{1}{\beta r}c(s,e) - \frac{1}{\beta}(h')^{-1}\left(\frac{y'(e)}{c_e(s,e)}\right)\right] =$$
$$= \frac{1}{\beta r}c_e(s,e) - \frac{1}{\beta}\left[h''\left((h')^{-1}\left(\frac{y'(e)}{c_e(s,e)}\right)\right)\right]^{-1}\frac{y''(e)c_e(s,e) - y'(e)c_{ee}(s,e)}{[c_e(s,e)]^2}$$

Since h is strictly convex, the properties of the functions $y(\cdot)$ and $c(\cdot,\cdot)$ imply that this partial derivative is strictly positive. Similarly, the partial derivative of the same expression with respect to s is strictly positive. Therefore, applying lemma 20, it follows that the subdifferential $\partial\overline{V}\left(\frac{1}{\beta r}c(s,e) - \frac{1}{\beta}(h')^{-1}\left(\frac{y'(e)}{c_e(s,e)}\right)\right)$ is increasing in e and s, and thus that the correspondence in (17) is strictly increasing in e and s (the latter because the subdifferential is strictly positive). It follows then by lemma 21 that e_s is strictly decreasing in s.

We mention here that if (ICE) binds, even under an additional assumption of differentiability of V, the signs of $\frac{dw_s}{ds}$ and $\frac{du_s}{ds}$ are generically undetermined. One can use the Implicit Function Theorem to derive the expression for $\frac{dw_s}{ds}$ from the system in (e_s, w_s) made of equations $y'(e_s) - V'(w_s)c_e(s,e_s) = 0$ and $h'\left(\frac{1}{r}c(s,e_s) - \beta w_s\right) + V'(w_s) = 0$, which can be obtained from the set of necessary first order conditions, but the sign of $\frac{dw_s}{ds}$ is undetermined. Similarly, one can derive a system of equations in (e_s, u_s) and conclude that the sign of $\frac{du_s}{ds}$ is undetermined. □

Proof of Lemma 6 Note that for any w and any $s \in [\underline{s}, \widehat{s})$, (ICE) binds if and only if $\beta w + u_s < c(s,e_s)$, where e_s is the solution to (16) or (17) for that particular value of s. Since (16) and (17) are independent of w, it follows that when (ICE) binds, e_s is constant in w. Moreover, as argued in the proof of proposition 5, u_s equals either 0 or $(h')^{-1}\left(\frac{y'(e_s)}{c_e(s,e_s)}\right)$, which since e_s is independent of w, implies that u_s is also independent of w. We conclude that the condition $\beta w + u_s < c(s,e_s)$

holds for all small enough w. Therefore, there exists $\widetilde{w}_s \equiv \inf \{w \,|\beta w + u_s \geq c\,(s, e_s)\,\}$, such that (ICE) binds for all $w \leq \widetilde{w}_s$.

On the other hand, as w increases immediately above $\widetilde{w}_s$, and thus (ICE) does not bind, e_s is determined by (15). Since the correspondence $H(w, e) \equiv -y'(e) + c_e\,(s, e)\,[-\partial V(w)]$ is increasing in w (because of the concavity of V), and strictly increasing in e (as we showed in the proof of proposition 5), we conclude that e_s is decreasing in w. As for u_s, if $u_s > 0$ and (ICE) does not bind (and thus $\mu_s = 0$), then from (9) and $\gamma \in -\partial V(w)$, it follows that $0 \in -h'(u_s) - \partial V(w)$. Since $H(u, w) \equiv -h'(u) - \partial V(w)$ is strictly decreasing in u and increasing w, it follows that u_s is increasing in w. Finally, when (ICE) does not bind, $w_s = w$ is strictly increasing in w. We conclude that once (ICE) does not bind for some value w, it will continue to be nonbinding for all higher values of w since, as w increases, $\beta w + u_s$ strictly increases, while $c\,(s, e_s)$ decreases because e_s does so. This completes the proof of the lemma 6. □

Proof of Proposition 7 We already argued in the proof of lemma 6 that whenever (ICE) binds, e_s and u_s are constant in w. Since (ICE) binding implies $w_s = \frac{1}{\beta r} c\,(s, e_s) - \frac{u_s}{\beta}$, w_s must be constant as well. Therefore, w_s is constant in w for $w \in [0, \widetilde{w}_s]$. When (ICE) does not bind, as also shown in the proof of claim 6, e_s is decreasing in w, and u_s is increasing in w, whereas, optimally, we have $w_s = w$, and thus w_s is strictly increasing in w. Moreover, since w_s is continuous in w, the fact that $w_s = w$ when $w = \widetilde{w}_s$ implies that on $[0, \widetilde{w}_s]$, it must be that $w_s = \widetilde{w}_s$. We examine now how w_s^n and u_s^n depend on w. First, w_s^n is strictly increasing since, optimally, $w_s^n = w$. Second, when $u_s^n > 0$, we have from (10), (12), and the fact that $w_s^n = w$, that $0 \in -h'(u_s^n) - \partial V(w)$, which implies then that u_s^n is increasing in w.

Next, we investigate how the threshold $\widehat{s}$ varies with w. Note that in the above we only showed that e_s is decreasing in w for $s \in [\underline{s}, \widehat{s})$, but in principle $\widehat{s}$ could still increase when w increases; we will argue now that this is not the case. We consider the case where the density $f(s)$ is continuous so that the Lagrangian $\mathcal{L}$ is continuously differentiable and the first order necessary condition has the standard form. The analysis of the general case is more tedious, but follows the same approach.

Employing thus the findings of propositions 3 and 4, $\mathcal{L}$ can be rewritten as

$$\begin{aligned}\mathcal{L} &= \int_{\underline{s}}^{\widehat{s}} \{y(e_s) + r[\beta V(w_s) - h(u_s)] + (1-r)[\beta V(w) - h(u^z)]\} f(s)ds \\ &+ \int_{\widehat{s}}^{\overline{s}} [\beta V(w) - h(u^z)] f(s)ds + \int_{\underline{s}}^{\widehat{s}} \mu_s [r(\beta w_s + u_s) - c(s, e_s)] f(s)ds + \\ &+ \gamma \left\{ \int_{\underline{s}}^{\widehat{s}} [r(\beta w_s + u_s) + (1-r)(\beta w + u^z) - c(s, e_s)] f(s)ds + \int_{\widehat{s}}^{\overline{s}} (\beta w + u^z) f(s)ds - w \right\}\end{aligned}$$

Since $\mathcal{L}$ is continuously differentiable as the integral of a continuous function, the necessary condition for the optimality of $\widehat{s}$, when $\widehat{s} \in (\underline{s}, \overline{s})$ is $\frac{\partial \mathcal{L}}{\partial \widehat{s}} = 0$. Employing the complementary slackness condition on constraint (ICE), $\frac{\partial \mathcal{L}}{\partial \widehat{s}} = 0$ becomes

$$y(e_{\widehat{s}}) + r[\beta V(w_{\widehat{s}}) - h(u_{\widehat{s}})] - r[\beta V(w) - h(u^z)] + \gamma\{[r(\beta w_{\widehat{s}} + u_{\widehat{s}}) - r(\beta w + u^z) - c(\widehat{s}, e_{\widehat{s}})]\} = 0 \tag{18}$$

Consider now for any w, any $s \in [\underline{s}, \widehat{s})$ with $u_s = 0$. By proposition 5, $u^z = 0$ as well. (18) becomes then

$$y(e_{\widehat{s}}) + r\beta[V(w_{\widehat{s}}) - V(w)] + \gamma[r\beta(w_{\widehat{s}} - w) - c(\widehat{s}, e_{\widehat{s}})] = 0 \tag{19}$$

If (ICE) does not bind, then $w_{\widehat{s}} = w$ and thus (19) becomes $y(e_{\widehat{s}}) - \gamma c(\widehat{s}, e_{\widehat{s}}) = 0$. Since $\gamma \in -\partial V(w)$, this can be rewritten as $0 \in H(w, \widehat{s}) \equiv -y(e_{\widehat{s}}) + c(\widehat{s}, e_{\widehat{s}})[-\partial V(w)]$. Since $e_{\widehat{s}}$ and $w_{\widehat{s}}$ are chosen optimally, by the Envelope Theorem, we have $\frac{\partial}{\partial \widehat{s}}\{-y(e_{\widehat{s}}) + c(\widehat{s}, e_{\widehat{s}})[-\partial V(w)]\} = -c_s(\widehat{s}, e_{\widehat{s}})\partial V(w_{\widehat{s}}) \subset (0, \infty)$. On the other hand, since V is concave, the correspondence $-\partial V(w)$ is increasing in w, and therefore so is $-y(e_{\widehat{s}}) + c(\widehat{s}, e_{\widehat{s}})[-\partial V(w)]$. We conclude the correspondence $H(w, \widehat{s})$ is strictly increasing in $\widehat{s}$ and increasing in w. This implies by lemma 21 that $\widehat{s}$ is decreasing in w as claimed by the proposition. If (ICE) binds, then $w_{\widehat{s}} = \frac{c(\widehat{s}, e_{\widehat{s}})}{\beta r}$ and (19) becomes $y(e_{\widehat{s}}) + r\beta\left[V\left(\frac{c(\widehat{s}, e_{\widehat{s}})}{\beta r}\right) - V(w)\right] + \gamma\left\{\left[r\beta\left(\frac{c(\widehat{s}, e_{\widehat{s}})}{\beta r} - w\right) - c(\widehat{s}, e_{\widehat{s}})\right]\right\} = 0$, which since $\gamma \in -\partial V(w)$ can be rewritten as $0 \in H(w, \widehat{s}) \equiv -y(e_{\widehat{s}}) + r\beta\left[-V\left(\frac{c(\widehat{s}, e_{\widehat{s}})}{\beta r}\right) + V(w) - w\partial V(w)\right]$. Then $c_s > 0$ and the fact that V strictly decreasing on $(0, \infty)$, where $\frac{c(\widehat{s}, e_{\widehat{s}})}{\beta r}$ lies, imply that $-V\left(\frac{c(\widehat{s}, e_{\widehat{s}})}{\beta r}\right)$ is strictly increasing in $\widehat{s}$ and thus immediately that $H(w, \widehat{s})$ is strictly increasing as well. On the other hand, let $x \in \partial V(w)$ and $x' \in \partial V(w')$, for some $w' > w$; since V is concave it must be that $x' \leq x$. Note then that we have $V(w) - V(w') \leq x'(w - w') \leq xw - x'w'$, where the first inequality follows from the definition of the superdifferential at w' of the concave function V and $x' \in \partial V(w')$ (see, for instance, Rockafellar (1970), pp. 214-215), while the second from $x' \leq x$. Therefore, $V(w') - w'x' \geq V(w) - wx$, which implies that the correspondence $V(w) - w\partial V(w)$ is increasing in w. It follows that $H(w, \widehat{s})$ is increasing in w, and thus by lemma 21 that $\widehat{s}$ is decreasing in w.

Finally, consider any value w for which $u_{\widehat{s}} > 0$. If (ICE) does not bind, by proposition 3, we have $u_s = u^z$, and therefore (18) becomes $y(e_{\widehat{s}}) - \gamma c(\widehat{s}, e_{\widehat{s}}) = 0$, where we also used the fact that $w_{\widehat{s}} = w$. As argued above, this implies that $\widehat{s}$ is decreasing in w. If (ICE) binds, then $w_{\widehat{s}} = \frac{c(\widehat{s}, e_{\widehat{s}})}{\beta r}$, and therefore employing also the facts that $\gamma \in -\partial V(w)$ and that $h'(u^z) = \gamma$ whenever $u^z > 0$, (18) becomes

$$0 \in H(w, \widehat{s}) \equiv -y(e_{\widehat{s}}) + r\beta \left[-V\left(\frac{c(\widehat{s}, e_{\widehat{s}})}{\beta r}\right) + V(w) - w\partial V(w)\right] + rh(u_{\widehat{s}}) + rA(w, \widehat{s}),$$

$$\text{where } A(w, \widehat{s}) \equiv \{t : t = -h(u^z) + u^z x, \text{ with } h'(u^z) = x \in -\partial V(w)\} \quad (20)$$

Since we have already showed above that $-y(e_{\widehat{s}}) + r\beta\left[-V\left(\frac{c(\widehat{s}, e_{\widehat{s}})}{\beta r}\right) + V(w) - w\partial V(w)\right]$ is strictly increasing in $\widehat{s}$ and increasing w, we aim to show that the correspondence $h'(u_{\widehat{s}}) + A(w, \widehat{s})$ has the same properties. Note that from (9) and (11) we have $h'(u_{\widehat{s}}) \in -\partial V\left(\frac{c(\widehat{s}, e_{\widehat{s}})}{\beta r}\right)$, while from (10) and (12), we have $h'(u^z) \in -\partial V(w)$ for any $\widehat{s}$. Therefore, $h(u_{\widehat{s}})$ is independent of w, while $A(w, \widehat{s})$ is independent of $\widehat{s}$. Since V is concave and $c_s > 0$, $h'(u_{\widehat{s}}) \in -\partial V\left(\frac{c(\widehat{s}, e_{\widehat{s}})}{\beta r}\right)$ implies immediately that $u_{\widehat{s}}$, and therefore $h(u_{\widehat{s}})$, are increasing in $\widehat{s}$. To complete the argument it is then enough to show that $A(w, \widehat{s})$ is increasing in w. Consider some arbitrary values $w' > w$, then arbitrary values $x' \in -\partial V(w')$ and $x \in -\partial V(w)$, and let $z' \equiv (h')^{-1}(x')$ and $z \equiv (h')^{-1}(x)$. We will show that $-h(z') + z'x' \geq -h(z) + zx$ which will imply that $A(w, \widehat{s})$ is indeed increasing in w. Since h is strictly convex, $h(z') - h(z) < h'(z')(z' - z)$, and so it is enough to show that $z'x' - zx \geq h'(z')(z' - z)$, i.e., that $z'[x' - h'(z')] \geq z[x - h'(z')]$. Since V is concave, $x' \geq x$, and thus $h'(z') \geq h'(z)$, which immediately imply that $x \leq h'(z')$. Together with $x' = h'(z')$, this implies that $z'[x' - h'(z')] = 0 \geq z[x - h'(z')]$. We conclude that $H(w, \widehat{s})$, as defined in (20), is strictly increasing in $\widehat{s}$ and increasing in w, and thus that $\widehat{s}$ is again decreasing in w. It is immediate then to argue that the same property holds for the treshold $\widehat{s}$ that maximizes $\mathcal{L}$ instead of $\widehat{\mathcal{L}}$. $\square$

Proof of Proposition 8 For the first part of the proposition note that for any continuation value w at the beginning of a given period, $\mathcal{P}$ can satisfy $\mathcal{A}$'s incentive constraints *in all future periods* with the following set of choice variables: $e_s = 0$, $w_s = w$, and $u_s = (1-\beta)\,w$ for all $s \in [\underline{s}, \overline{s}]$. These correspond to $\mathcal{A}$ being retired and delivered utility $(1-\beta)\,w$ in every period for perpetuity. The discounted present value of the contract to $\mathcal{P}$ corresponding to this set of choice variables is $-\frac{1}{1-\beta}h((1-\beta)\,w)$. Therefore, if V solves problem (2)-(3), then it must be that $V(w) \geq -\frac{1}{1-\beta}h((1-\beta)\,w)$ for all w.

For the second part of the statement in *(i)*, note that if, optimally, $e_s = 0$ for all $s \in [\underline{s}, \overline{s}]$, then $\mathcal{P}$'s problem is that of maximizing the objective function in (2) subject to the participation constraint in (PKC) and the non-negativity of the choice variables. By propositions 3 and 4, in the corresponding solution, we have $w_s = w$ for all s and $u_s = u^z$ for all s and some $u^z \geq 0$. Therefore, the binding (PKC) becomes $\beta w + u^z = w$, i.e., $u^z = (1-\beta)\,w$. Therefore, $V(w) = -h((1-\beta)\,w) + \beta V(w)$, implying indeed that $V(w) = -\frac{1}{1-\beta}h((1-\beta)\,w)$.

For part *(ii)*, we only need to examine the case where $\overline{\overline{w}}$ is finite. First, since $V(w) \geq -\frac{1}{1-\beta}h((1-\beta)\,w)$, $V(\overline{\overline{w}}) = -\frac{1}{1-\beta}h((1-\beta)\,\overline{\overline{w}})$, and $-\frac{1}{1-\beta}h((1-\beta)\,w)$ is concave and differentiable in w, employing Theorem 4.10 from Stokey and Lucas (1989), it follows that V is differentiable at $\overline{\overline{w}}$ and that $V'(\overline{\overline{w}}) = -h'((1-\beta)\,\overline{\overline{w}})$. Next, we will show that $e_s = 0$ for all $s \in [\underline{s}, \overline{s}]$ at $\overline{\overline{w}}$. To this aim, assume by contradiction that at $\overline{\overline{w}}$, it is optimal instead to have $e_s > 0$ for all $s \in [\underline{s}, \widehat{s})$, for some $\widehat{s} \in [\underline{s}, \overline{s}]$ with $F(\widehat{s}) > 0$. Let then (S_1, S_2) be the partition of $[\underline{s}, \widehat{s}]$ such that (ICE) binds for all $s \in S_1$ and does not bind for all $s \in S_2$, and denote $m(S_i) \equiv \int_{s \in S_i} f(s)ds$. By propositions 3 and 4, we have then $w_s = \overline{\overline{w}}$ and $u_s = u^z$ for all $s \in [\widehat{s}, \overline{s}] \cup S_2$, and $w_s^n = \overline{\overline{w}}$ and $u_s^n = u^z$ for all $s \in [\underline{s}, \overline{s}]$. In addition, $r\,(\beta w_s + u_s) - c\,(s, e_s) = 0$ for all $s \in S_1$. Employing these, it follows that the binding condition in (PKC) becomes $\left(\beta\overline{\overline{w}}+u^z\right)\left[(1-r)\,m(S_1)+m(S_2)+1-F(\widehat{s})\right]-\int_{s\in S_2}c(s,e_s)f(s)ds=\overline{\overline{w}}$. Therefore, $u^z=\frac{1}{(1-r)m(S_1)+m(S_2)+1-F(\widehat{s})}\left[\left\{1-\beta\left[(1-r)\,m(S_1)+m(S_2)+1-F(\widehat{s})\right]\right\}\overline{\overline{w}}+\int_{s\in S_2}c(s,e_s)f(s)ds\right]>0$. Since $F(\widehat{s})>0$, we have then that $u^z>(1-\beta)\,\overline{\overline{w}}$, which since h is strictly convex implies $h'(u^z)>h'((1-\beta)\,\overline{\overline{w}})$. However, conditions (10) and (12) imply that at optimum, we must have $h'(u^z)\in-\partial V(\overline{\overline{w}})$. As argued above, $\partial V(\overline{\overline{w}})=\{-h'((1-\beta)\,\overline{\overline{w}})\}$, and thus $h'(u^z)=h'((1-\beta)\,\overline{\overline{w}})$. This provides the desired contradiction. Finally, since by proposition 7, e_s is decreasing in w, it must be that $e_s=0$ for all $s\in[\underline{s},\overline{s}]$ and $w\geq\overline{\overline{w}}$. □

Proof of Proposition 9 Using the multipliers from the definition of $\mathcal{L}$, the per-period change in $\mathcal{P}$'s payoff from a small increase in r is $\Delta V=\int_{\underline{s}}^{\widehat{s}}\left\{\left[\beta V(w_s)-h\left(u_s\right)\right]-\left[\beta V\left(w_s^n\right)-h\left(u_s^n\right)\right]\right\}f(s)ds+\int_{\underline{s}}^{\widehat{s}}\mu_s\left(\beta w_s+u_s\right)f(s)ds+\gamma\int_{\underline{s}}^{\widehat{s}}\left[\left(\beta w_s+u_s\right)-\left(\beta w_s^n+u_s^n\right)\right]f(s)ds$, which can be written as $\Delta V=\int_{\underline{s}}^{\widehat{s}}\Delta V_s f(s)ds$, where

$$\Delta V_s\equiv\beta\left[V(w_s)+\left(\gamma+\mu_s\right)w_s-V\left(w_s^n\right)-\gamma w_s^n\right]+\left[h\left(u_s^n\right)-\gamma u_s^n-h\left(u_s\right)+\left(\gamma+\mu_s\right)u_s\right] \tag{21}$$

We will argue that both terms from (21) are nonnegative. First, note that since $\gamma+\mu_s\in-\partial V(w_s)$ and $\gamma\in-V\left(w_s^n\right)$, while by proposition 3 we have $w_s\geq w_s^n$, to show that the first term is nonnegative, it would be enough for the correspondence $V(w)-w\partial V(w)$ to be increasing, which has already been shown in the proof of proposition 7. As for the second term in (21), note first that if $u_s=0$, then by proposition 3, $u_s^n=0$ as well and thus the term is 0. If $u_s>0$, then from (9) it follows that $\gamma+\mu_s=h'\left(u_s\right)$, while from (10), we have $\gamma\leq h'(u_s^n)$. It is enough then to show that $\left[h\left(u_s^n\right)-h'\left(u_s^n\right)u_s^n\right]-\left[h\left(u_s\right)-h'\left(u_s\right)u_s\right]\geq0$. Since by proposition 3, we have $u_s\geq u_s^n$, it is enough to show that the function $h\left(u\right)-h'\left(u\right)u$ is decreasing, which follows immediately by taking its derivative and accounting for the fact that h is convex. We conclude that, indeed, $\Delta V_s\geq0$, and thus $\Delta V\geq0$. Since the per-period change to $\mathcal{P}$'s payoff from a small increase in r is positive, it follows that $V\left(w\right)$, which is the discounted sum of future period payoffs, is increasing in r.[19] □

[19] This argument is similar to the one employed in the proof of lemma 1 in Kovrijnykh (2013).

Proof of Proposition 1 The recursive representation of the optimal contract problem is

$$V(w) = \max_{\{e_s,w_s,u_s\}_{s\in[\underline{s},\overline{s}]}} \int_{\underline{s}}^{\overline{s}} \{y(e_s) + r[\beta V(w_s) - h(u_s)]\} f(s)ds \tag{22}$$

$$\text{s.t.} \int_{\underline{s}}^{\overline{s}} [r(\beta w_s + u_s) - c(s,e_s)] f(s)ds \geq w \tag{23}$$

The Lagrangian is $\mathcal{L} \equiv \{y(e_s) + r[\beta V(w_s) - h(u_s)]\} f(s) + \gamma \{[r(\beta w_s + u_s) - c(s,e_s)] f(s) - w\}$, with $\gamma \in \mathbb{R}_+$, and so the necessary first order conditions, after cancelling out terms, are

$$\frac{\partial \mathcal{L}}{\partial u_s} = -h'(u_s) + \gamma \leq 0, \text{ and } = 0 \text{ if } u_s > 0 \tag{24}$$

$$\frac{\partial \mathcal{L}}{\partial e_s} = 0 \Rightarrow y'(e_s) - \gamma c_e(s,e_s) \leq 0, \text{ and } = 0 \text{ if } e_s > 0 \tag{25}$$

$$\frac{\partial \mathcal{L}}{\partial w_s} = 0 \Rightarrow 0 \in \partial V(w_s) + \gamma \text{ if } w_s > 0;\ \frac{\partial \mathcal{L}}{\partial w_s} \leq 0 \Rightarrow V'(w_s) + \gamma \leq 0 \text{ if } w_s = 0 \tag{26}$$

From (24), it follows that there exists $u^0 \in \mathbb{R}_+$ such that for all $s \in [\underline{s},\overline{s}]$, with $u_s > 0$, we have $u_s = u^0$. Assume now by contradiction that $u^0 > 0$ and that $u_s = 0$ for some $s \in [\underline{s},\overline{s}]$. From (24) it follows then that $-h'(0) + \gamma \leq 0$, which since also from (24), we have $\gamma = h'(u^0)$, implies that $-h'(0) + h'(u^0) \leq 0$. This contradicts the fact that h is strictly convex. Therefore, it must be that $u_s = u^0$ for all $s \in [\underline{s},\overline{s}]$, with u^0 potentially being equal to 0. Similarly, from (26), using the fact that V is concave (because $\mathcal{P}$ can offer lotteries over continuation values), it follows that there exists $w^0 \geq 0$ such that it is optimal that for all $s \in [\underline{s},\overline{s}]$ to set $w_s = w^0$.[20] Using the Envelope Theorem on problem (22)-(23), we have $\gamma \in \frac{\partial \mathcal{L}}{\partial w} = -\partial V(w)$, which combined with (26), implies that it is optimal to set $w^0 = w$. Therefore, in the first period of the contract, when the value promised to the agent is 0, it is optimal to set $w^0 = 0$, implying that at the beginning of the second stage, the agent is again promised a value equal to 0. By induction, he is promised a value 0 at the beginning of every period. Since he must consume, it follows that $u^0 > 0$. Finally, condition (1) from the text of the proposition follows immediately from (24) and (25). □

[20] However, if V is constant on some interval $[0,\underline{w}]$, then it is only *sufficient*, but not necessary for optimality to have w_s constant for all s. The necessary condition is that either $w_s = w^0 > \underline{w}$ for all s, or $w_s \in [0,\underline{w}]$ for all s.

Appendix A2. Existence of an Optimal Contract.

A solution to problem (2)-(3) can be guaranteed to exist only when the set of "candidate" incentive compatible contracts is compact. By the Arzela-Ascoli Theorem, a set of functions is compact if and only if it is pointwise bounded, closed and equicontinuous. We will argue below that one can restrict attention to contracts which are pointwise bounded in the process of identifying the optimal contract, while the closedness of the set will follow from the continuity of the functions involved. When X and Y are two metric spaces, a family of functions $\mathcal{F}$ is said to be *equicontinuous at a point* $x_0 \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(f(x_0), f(x)) < \varepsilon$ for all $f \in \mathcal{F}$ and all x such that $d(x_0, x) < \delta$. The family is *equicontinuous* if it is equicontinuous at each point of X. We will argue below that one can guarantee the existence of a solution to problem (2)-(3) in situations where the set of *admissible* incentive compatible contracts $\{e_s, w_s, w_s^n, u_s, u_s^n\}_{s\in[\underline{s},\overline{s}]}$ is known to be a family which is equicontinuous. One situation when this condition is satisfied immediately is when the support of s is finite. Another situation is when the contracts from this set are Lipschitz continuous with a *common* Lipschitz constant.

Define the function G as

$$G\left(w, \{w_s, w_s^n\}_{s\in[\underline{s},\overline{s}]}\right) \equiv \max_{\{e_s,u_s,u_s^n\}_{s\in[\underline{s},\overline{s}]}} \int_{\underline{s}}^{\overline{s}} \left[y\left(e_s\right) - rh\left(u_s\right) - (1-r)\,h\left(u_s^n\right)\right] f(s)ds \tag{27}$$

$$\text{subject to: } r\left(\beta w_s + u_s\right) - c\left(s, e_s\right) \geq 0, \text{ for all } s \in [\underline{s}, \overline{s}] \tag{28}$$

$$\int_{\underline{s}}^{\overline{s}} \left[r\left(\beta w_s + u_s\right) + (1-r)\left(\beta w_s^n + u_s^n\right) - c\left(s, e_s\right)\right] f(s)ds \;\geq w \tag{29}$$

$$e_s \geq 0,\ w_s \geq 0,\ w_s^n \geq 0,\ u_s \geq 0,\ u_s^n \geq 0, \text{ for all } s \in [\underline{s}, \overline{s}] \tag{30}$$

Also, define the correspondence $\Gamma\left(w, \{w_s, w_s^n\}_{s\in[\underline{s},\overline{s}]}\right)$ to be the set of all functions $\{e_s, u_s, u_s^n\}_{s\in[\underline{s},\overline{s}]}$ that satisfy constraints (28)-(30). Note the constraints from the definition of G are slightly different than those from problem (2)-(3).

Lemma 22 *The function G is well defined, continuous in all arguments, bounded and concave.*

The correspondence Γ *is convex-valued.*

Proof. Proof. Note first that in problem (27)-(30) we can restrict attention to $e_s \in [0, \overline{e}]$, where $\overline{e}$ satisifies $y(\overline{e}) = h(c(\underline{s}, \overline{e}))$ since in order to satisfy (30) when e_s increases above $\overline{e}$, $rh(u_s) + (1-r)h(u_s^n)$ must increase by more than the corresponding increase in $y(e_s)$, on net lowering the value of the objective function. Second, we can also restrict attention in the same problem to profiles of induced utilities satisfying $u_s, u_s^n \in \left[0, \frac{1}{\min\{r,1-r\}}[w + c(s, e_s)]\right]$ a.e. since setting some of these utilities higher has no effect on (28), as this constraint is already satisfied, but because of the convexity of h, lowers the value of the objective function.[21] Under the assumption of an equicontinuous set of admissible contracts, by the continuity of the functions involved and the boundedness of $\{e_s, u_s, u_s^n\}_{s\in[\underline{s},\overline{s}]}$, it follows that G is well defined, i.e., that the maximum is attained. Moreover, employing the Theorem of the Maximum (see, for instance, Theorem 3.6 in Stokey and Lucas (1989), or page 306 in Ok (2007) for a statement that allows for the maximizers and parameters to take values in any metric spaces[22]), it follows that G is continuous in all variables. Finally, G is bounded from above since $y(e_s)$ is bounded, and is bounded from below by 0.

Next, we will show that $\Gamma\left(w, \{w_s, w_s^n\}_{s\in[\underline{s},\overline{s}]}\right)$ is convex-valued. Thus, assume that $\chi^1 \equiv \left\{e_s^1, u_s^1, u_s^{n1}\right\}_{s\in[\underline{s},\overline{s}]}$ and $\chi^2 \equiv \left\{e_s^2, u_s^2, u_s^{n2}\right\}_{s\in[\underline{s},\overline{s}]}$ are in $\Gamma\left(w, \{w_s, w_s^n\}_{s\in[\underline{s},\overline{s}]}\right)$. Clearly (30) is satisfied for any $\alpha\chi^1 + (1-\alpha)\chi^2$ with $\alpha \in [0,1]$. We will show next that (28) is also satisfied. Thus, note that $r\left[\beta w_s + \alpha u_s^1 + (1-\alpha)u_s^2\right] \geq \alpha c\left(s, e_s^1\right) + (1-\alpha)c\left(s, e_s^2\right) > c\left(s, \alpha e_s^1 + (1-\alpha)e_s^2\right)$ with the first inequality following from the fact that χ^1 and χ^2 satisfy (28), while the second inequality

[21] To see this, consider the simple case where s is uniformly distributed on $[\underline{s}, \overline{s}]$ and assume, for instance, that $u_s > \frac{1}{\min\{r,1-r\}}[w + c(s, e_s)]$ for all s in a set of strictly positive measure. In order for (29) to be satisfied, there must also exist some set of strictly positive measure such that for all s' in this set, we have $u_{s'} < \frac{1}{\min\{r,1-r\}}[w + c(s', e_{s'})]$. One can then decrease u_s by some small amount, while increasing $u_{s'}$ by the same amount so as to keep (29) satisfied. Since (28) is also satisfied, because of the convexity of h, this adjustment leads to a strictly higher value of the objective function. A similar type of adjustment can also be made in the case of an arbitrary distribution of s.

[22] Here, these metric spaces are those of bounded functions on $[\underline{s}, \overline{s}]$ with the metric $d_\infty(f, g) \equiv \sup_{s\in[\underline{s},\overline{s}]} |f(s) - g(s)|$. Provided that the existence of a maximum would be guaranteed in a case with a generic distribution f, the rest of the argument would go through.

is given by $c_{ee} > 0$. To show that (29) is satisfied, note that we have

$$\int_{\underline{s}}^{\overline{s}} \left[r\left(\beta w_s + \alpha u_s^1 + (1-\alpha)\, u_s^2\right) + (1-r)\left(\beta w_s^n + \alpha u_s^{n1} + (1-\alpha)\, u_s^{n2}\right) - c\left(s, \alpha e_s^1 + (1-\alpha)\, e_s^2\right)\right] f(s) ds \geq$$

$$\geq \alpha \int_{\underline{s}}^{\overline{s}} \left[r\left(\beta w_s + \alpha u_s^1\right) + (1-r)\left(\beta w_s^n + \alpha u_s^{n1}\right) - c\left(s, \alpha e_s^1\right)\right] f(s) ds +$$
$$+ (1-\alpha) \int_{\underline{s}}^{\overline{s}} \left[r\left(\beta w_s + \alpha u_s^2\right) + (1-r)\left(\beta w_s^n + \alpha u_s^{n2}\right) - c\left(s, \alpha e_s^2\right)\right] f(s) ds \geq w$$

with the first inequality following from $\alpha c\left(s, e_s^1\right) + (1-\alpha)\, c\left(s, e_s^2\right) > c\left(s, \alpha e_s^1 + (1-\alpha)\, e_s^2\right)$, and the second from the fact that χ^1 and χ^2 satisfy (29).

The concavity of G follows by the standard argument in the proof of the Theorem of the Maximum under Convexity (see Theorem 9.17 in Sundaram (1996)). □

As shown in the proof of proposition 8, we have $V(w) \geq -\frac{1}{1-\beta} h((1-\beta)\, w)$ for all w. Let $S(w) \equiv V(w) + \frac{1}{1-\beta} h((1-\beta)\, w)$ and note that $0 \leq S(w) \leq S^0 \equiv \frac{1}{1-\beta} \int_{\underline{s}}^{\overline{s}} \left[y\left(e_s^0\right) - c\left(s, e_s^0\right)\right] f(s) ds$, where S^0 is the present value of the sum of first-best total payoffs from all future periods.

Let $\mathcal{M} \equiv \{S_g : \mathbb{R}_+ \to \mathbb{R}_+ \colon S_g$ is continuous, concave, and satisfies $S_g(w) \geq 0$ for all $w \in \mathbb{R}_+\}$ and define on $\mathcal{M}$ the operator

$$T_g(S_g)\,(w) \equiv \frac{1}{1-\beta} h((1-\beta)\, w) + \\ + \max_{\{w_s, w_s^n\}_{s\in[\underline{s},\overline{s}]}} \left\{ G\left(w, \{w_s, w_s^n\}_{s\in[\underline{s},\overline{s}]}\right) + \beta \int_{\underline{s}}^{\overline{s}} \left[r V_g(w_s) + (1-r)\, V_g\left(w_s^n\right)\right] f(s) ds \right\} \quad (31)$$

where $V_g\,(w) \equiv S_g(w) - \frac{1}{1-\beta} h((1-\beta)\, w)$. Since G is bounded and continuous in all arguments, while V_g is continuous, bounded from above and satisfies $\lim_{w\to\infty} V_g\,(w) = \lim_{w\to\infty} \left[S_g(w) - \frac{1}{1-\beta} h((1-\beta)\, w)\right] = -\infty$ (because S_g is bounded), it follows that the maximizer from (31) is attained (we are employing here again the additional assumption that the set of admissible contracts is equicontinuous). To see this, note that since the objective function is continuous, if the maximizer was not attained, the objective function would be unbounded. Because of the continuity of the function, this can happen only as one of the arguments goes approaches ∞; however, in that case the value of the objective function actually approaches $-\infty$. We conclude that $T_g(S_g)(w)$ is well defined. Define now the following operator on $\mathcal{M}$

$$T(S_g)(w) \equiv \min \left\{ cav\left(T_g(S_g)(w)\right), S^0 \right\} \tag{32}$$

where by $cav\left(T_g(S_g)(w)\right)$ we mean the concave envelope of the function $T_g(S_g)(w)$.

Lemma 23 *The operator T has a fixed point S which is continuous in w.*

Proof. By the Theorem of the Maximum under Convexity, $T_g(S_g)(w)$ is continuous and therefore T has this property as well. The concavity and boundedness of $T(S_g)(w)$ are immediate. Therefore, T maps $\mathcal{M}$ into $\mathcal{M}$. The operator T_g clearly satisfies the Blackwell Sufficient Conditions (see Theorem 3.3 in Stokey and Lucas (1989)). Moreover, T satisfies these conditions as well. To see this, note first that the monotonicity property is immediate. For the discounting property, note that $T_g(S_g+a)(w) - T_g(S_g)(w) \geq \min\left\{T_g(S_g+a)(w), S^0\right\} - \min\left\{T_g(S_g)(w), S^0\right\}$ whenever $a \geq 0$. Therefore, since the fact that T_g satsifies discounting implies that $\beta a \geq T_g(S_g+a)(w) - T_g(S_g)(w)$, we also have $\beta a \geq T(S_g+a)(w) - T(S_g)(w)$, implying that T satisfies discounting as well. Applying the Contraction Mapping Theorem to T (see Theorem 3.2 in Stokey and Lucas (1989)), it follows that T has a unique fixed point S, which is continuous and concave in w. □

This fixed point of T will then deliver the solution to problem (2)-(3). To see this, note that problem (2)-(3) can be written as $S(w) = T(S)(w)$ with several differences determined by the specific constraints from the definition of the function G in (27)-(30) which aided above in proving that $\Gamma\left(w, \{w_s, w_s^n\}_{s\in[\underline{s},\overline{s}]}\right)$ is a convex-valued correspondence and that G is concave. Thus, first, we did *not* incorporate the insight that optimally it must be that $e_s = 0$ for all $s \in [\widehat{s}, \overline{s}]$ for some value $\widehat{s}$. Second, we imposed that $r(\beta w_s + u_s) - c(s, e_s) \geq 0$ must hold for all $s \in [\underline{s}, \overline{s}]$, including the values of s with $e_s = 0$; these additional constraints are implied by (30), since $c(s, 0) = 0$, and therefore they do not change the constraint set. Third, we restricted attention to functions $\{w_s, w_s^n\}_{s\in[\underline{s},\overline{s}]}$ which are bounded because G is bounded, while $\lim\limits_{w\to\infty} V(w) \leq \lim\limits_{w\to\infty}\left[S^0 - \frac{1}{1-\beta}h((1-\beta)w)\right] = -\infty$.

Problems $S(w) = T(S)(w)$ and (2)-(3) are thus equivalent.

Letting $V(w) \equiv S(w) - \frac{1}{1-\beta} h((1-\beta) w)$, by the standard argument from Theorem 9.2 in Stokey and Lucas (1989), it follows that V is a solution to problem (2)-(3). Moreover, $V(w)$ is also continuous and concave in w, the latter because S is concave and h is convex. $\square$

Appendix A3. Proofs for Section 4.

To solve $\mathcal{P}$'s problem, we employ methods from optimal control theory. To this aim, we write the principal's problem as an optimal control problem with control variables $x_s \equiv e'_s$, $t_s \equiv w^{n\prime}_s$, $q_s \equiv u'_s$, $k_s \equiv u^{n\prime}_s$, and state variables w_s, w^n_s, u_s, u^n_s, and e_s. To account for the promise keeping constraint (PKC), we introduce a new state variable

$$v_s \equiv \int_{\underline{s}}^{s} \left\{ r\left(\beta w_\sigma + u_\sigma\right) + (1-r)\left(\beta w^n_\sigma + u^n_\sigma\right) - c\left(\sigma, e_\sigma\right) \right\} f(\sigma) d\sigma \tag{33}$$

and rewrite (PKC) as the transversality condition $v_{\overline{s}} \geq w$. The other transversality condition on v is $v_{\underline{s}} = 0$. There are no transversality conditions on the remaining state variables.

The optimal control problem is then

$$\max_{\{x_s, t_s, q_s, k_s\}_{s \in [\underline{s}, \overline{s}]}} \int_{\underline{s}}^{\overline{s}} \left\{ y\left(e_s\right) + r\left[\beta V(w_s) - h\left(u_s\right)\right] + (1-r)\left[\beta V\left(w^n_s\right) - h\left(u^n_s\right)\right] \right\} f(s) ds \tag{34}$$

$$\text{s.t. } e'_s = x_s;\ w^{n\prime}_s = t_s;\ u'_s = q_s;\ u^{n\prime}_s = k_s \tag{35}$$

$$w'_s = -\frac{1}{\beta} q_s - \frac{1-r}{r}\left(t_s + \frac{1}{\beta} k_s\right) + \frac{1}{\beta r} c_e(s, e_s) x_s \tag{36}$$

$$v'_s = \left[r\left(\beta w_s + u_s\right) + (1-r)\left(\beta w^n_s + u^n_s\right) - c\left(s, e_s\right)\right] f(s) \tag{37}$$

$$r\left(\beta w_{\overline{s}} + u_{\overline{s}}\right) + (1-r)\left(\beta w^n_{\overline{s}} + u^n_{\overline{s}}\right) - c\left(\overline{s}, e_{\overline{s}}\right) - (1-r)\left(\beta w^n_s + u^n_s\right) \geq 0, \text{ for all } s \in [\underline{s}, \overline{s}] \tag{38}$$

$$v_{\underline{s}} = 0;\ v_{\overline{s}} \geq w \tag{39}$$

$$w_s \geq 0;\ w^n_s \geq 0;\ u_s \geq 0;\ u^n_s \geq 0, \text{ for all } s \in [\underline{s}, \overline{s}] \tag{40}$$

where (36) follows from (ICTW), and (38) from (ICEW). Conditions (38) and (40) are so-called pure state constraints in optimal control problems, while conditions in (39) are boundary constraints.

The Hamiltonian associated with problem (34)-(40) is

$$
\begin{aligned}
H(\cdot,s) \equiv {} & \left\{y(e_s) + r\left[\beta V(w_s) - h(u_s)\right] + (1-r)\left[\beta V(w_s^n) - h(u_s^n)\right]\right\} f(s) + \\
& + \lambda_s^1 x_s + \lambda_s^2 \left[-\frac{1}{\beta} q_s - \frac{1-r}{r}\left(t_s + \frac{1}{\beta} k_s\right) + \frac{1}{\beta r} c_e(s,e_s) x_s\right] + \\
& + \lambda_s^3 \left[r\left(\beta w_s + u_s\right) + (1-r)\left(\beta w_s^n + u_s^n\right) - c(s,e_s)\right] f(s) + \lambda_s^4 t_s + \lambda_s^5 q_s + \lambda_s^6 k_s
\end{aligned}
$$

while the Lagrangian that accounts for the state constraints is

$$
\begin{aligned}
L(\cdot,s) \equiv {} & H(\cdot,s) + \gamma_s^1 \left[r\left(\beta w_{\overline{s}} + u_{\overline{s}}\right) + (1-r)\left(\beta w_{\overline{s}}^n + u_{\overline{s}}^n\right) - c\left(\overline{s}, e_{\overline{s}}\right) - (1-r)\left(\beta w_s^n + u_s^n\right)\right] f(s) + \\
& + \left(\gamma_s^2 w_s + \gamma_s^3 w_s^n + \gamma_s^4 u_s + \gamma_s^5 u_s^n\right) f(s)
\end{aligned}
$$

where $\left\{\lambda_s^i\right\}_{i\in\overline{1,6}}$ and $\left\{\gamma_s^i\right\}_{i\in\overline{1,5}}$ are functions defined on $[\underline{s},\overline{s}]$. Since the Lagrangian is linear in all control variables, while the domain of these variables is unbounded, a solution to this problem necessarily involves a so-called *singular control*, i.e., it must satisfy $\frac{\partial H}{\partial j} = 0$ for all $j \in \{x_s, t_s, q_s, k_s\}$ and all $s \in [\underline{s},\overline{s}]$. By Pontryagin's Maximum Principle, which provides the necessary first order conditions in optimal control problems, there exist almost everywhere differentiable functions $\left\{\lambda_s^i\right\}_{i\in\overline{1,6}}$ and almost everywhere continuous functions $\left\{\gamma_s^i\right\}_{i\in\overline{1,5}}$ such that conditions (41)-(50) below are satisfied almost everywhere.

$$
\frac{\partial H}{\partial x_s} = \lambda_s^1 + \lambda_s^2 \frac{1}{r\beta} c_e(s,e_s) = 0;\ \frac{\partial H}{\partial t_s} = -\frac{1-r}{r}\lambda_s^2 + \lambda_s^4 = 0 \tag{41}
$$

$$
\frac{\partial H}{\partial q_s} = -\frac{1}{\beta}\lambda_s^2 + \lambda_s^5 = 0;\ \frac{\partial H}{\partial k_s} = -\frac{1}{\beta}\frac{1-r}{r}\lambda_s^2 + \lambda_s^6 = 0 \tag{42}
$$

$$\lambda_s^{1\prime} = -\frac{\partial L}{\partial e_s} = \left[-y'\left(e_s\right) + \lambda_s^3 c_e\left(s, e_s\right)\right] f(s) - \lambda_s^2 \frac{1}{r\beta} c_{ee}\left(s, e_s\right) x_s \tag{43}$$

$$\lambda_s^{2\prime} = -\frac{\partial L}{\partial w_s} = r\beta\left[-V'(w_s) - \lambda_s^3 - \gamma_s^2\right] f(s);\ \lambda_s^{3\prime} = -\frac{\partial L}{\partial v_s} = 0 \tag{44}$$

$$\lambda_s^{4\prime} = -\frac{\partial L}{\partial w_s^n} = (1-r)\,\beta\left[-V'\left(w_s^n\right) - \lambda_s^3 + \gamma_s^1 - \gamma_s^3\right] f(s) \tag{45}$$

$$\lambda_s^{5\prime} = -\frac{\partial L}{\partial u_s} = r\left[h'(u_s) - \lambda_s^3 - \gamma_s^4\right];\ \lambda_s^{6\prime} = -\frac{\partial L}{\partial u^n} = r\left[h'(u_s^n) - \lambda_s^3 + \gamma_s^1 - \gamma_s^5\right] \tag{46}$$

In addition, denoting $\Lambda \equiv \int_{\underline{s}}^{\overline{s}} L(\cdot, s) f(s) ds$ and $\Gamma \equiv \int_{\underline{s}}^{\overline{s}} \gamma_s^1 f(s) ds$, the following must hold.

$$\lambda_{\underline{s}}^i = 0, \text{ for } i \in \{1,2,4,5,6\};\ \lambda_{\underline{s}}^3 \in \mathbb{R};\ \lambda_{\overline{s}}^3 \in \mathbb{R}_+ \tag{47}$$

$$\lambda_{\overline{s}}^1 = \frac{\partial \Lambda}{\partial e_{\overline{s}}} = -c_e\left(\overline{s}, e_{\overline{s}}\right)\Gamma;\ \lambda_{\overline{s}}^2 = \frac{\partial \Lambda}{\partial w_{\overline{s}}} = r\beta\Gamma;\ \lambda_{\overline{s}}^4 = \frac{\partial \Lambda}{\partial w_{\overline{s}}^n} = (1-r)\,\beta\Gamma;\ \lambda_{\overline{s}}^5 = \frac{\partial \Lambda}{\partial u_{\overline{s}}} = r\Gamma;\ \lambda_{\overline{s}}^6 = \frac{\partial \Lambda}{\partial u_{\overline{s}}^n} = (1-r)\Gamma \tag{48}$$

$$\gamma_s^1 \geq 0, \text{ and } = 0 \text{ if } r\left(\beta w_{\overline{s}} + u_{\overline{s}}\right) + (1-r)\left(\beta w_{\overline{s}}^n + u_{\overline{s}}^n\right) - c\left(\overline{s}, e_{\overline{s}}\right) - (1-r)\left(\beta w_s^n + u_s^n\right) > 0 \tag{49}$$

$$\gamma_s^2 \geq 0, \text{ and } = 0 \text{ if } w_s > 0;\ \gamma_s^3 \geq 0, \text{ and } = 0 \text{ if } w_s^n > 0 \tag{50}$$

$$\gamma_s^4 \geq 0, \text{ and } = 0 \text{ if } u_s > 0;\ \gamma_s^5 \geq 0, \text{ and } = 0 \text{ if } u_s^n > 0 \tag{51}$$

The following claim, based on the Arrow Sufficiency Theorem, states the sufficiency of the conditions in (41)-(51) for the problem (34)-(40) and the uniqueness of the solution.

Claim 24 *If the trajectory* $\{e_s, w_s, u_s, w_s^n, u_s^n, v_s, x_s, t_s, q_s, k_s\}_{s\in[\underline{s},\overline{s}]}$ *satisfies (35)-(40) and (41)-(51) with costate variables* $\left\{\{\lambda_s^i\}_{i\in\overline{1,6}}\right\}$*, then it is the unique solution to (34)-(40).*

Proof of Claim 24 The proof follows the strategy of a similar result from Barbos (2016). We employ the Arrow Sufficiency Theorem for optimal control problems with mixed constraints (see Theorem 6.4 on page 166 in Caputo (2005)). The maximized Hamiltonian evaluated at the costate functions $\{\lambda_s^i\}_{i\in\overline{1,6}}$ (but not necessarily at the solution $\{e_s, w_s, u_s, w_s^n, u_s^n, v_s\}$) equals $\{y\left(e_s\right) + r\left[\beta V(w_s) - h\left(u_s\right)\right] + (1-r)\left[\beta V\left(w_s^n\right) - h\left(u_s^n\right)\right]\} f(s) +$

$+\lambda_s^3 \left[r\left(\beta w_s + u_s\right) + (1-r)\left(\beta w_s^n + u_s^n\right) - c\left(s, e_s\right)\right] f(s)$ by (41) and (42).[23] From (44) and (47), it follows that λ_s^3 is a nonnegative constant, which we denote by λ^3. Therefore the maximized Hamiltonian is concave in $\{e_s, w_s, u_s, w_s^n, u_s^n, v_s\}$ and strictly concave in $\{e_s, u_s, u_s^n\}$ by the assumptions imposed on $y(\cdot)$, $h(\cdot)$ and $c(\cdot, \cdot)$. The Arrow Sufficiency Theorem implies then that the necessary conditions in (41)-(51) are sufficient for problem (34)-(40). The theorem, as stated in Caputo (2005), requires strict concavity of the maximized Hamiltonian in the state variables for the uniqueness of the solution. By following its proof, it is evident that since the maximized Hamiltonian is strictly concave in $\{e_s, u_s, u_s^n, w_s, w_s^n\}$, then these variables must be unique in any solution to (34)-(40). The uniqueness of $\{v_s\}$ follows from its definition in (33), while the uniqueness of the controls follows from their definitions whenever the corresponding state variables are unique. □

Proof of Proposition 13 For part *(i)*, differentiating $-\frac{1-r}{r}\lambda_s^2 + \lambda_s^4 = 0$ with respect to s and then substituting $\lambda_s^{2\prime}$ and $\lambda_s^{4\prime}$ from (44) and (45), we obtain

$$V'(w_s^n) - V'(w_s) = \gamma_s^1 + \gamma_s^2 - \gamma_s^3 \tag{52}$$

Now, if $w_s^n > 0$, then $\gamma_s^3 = 0$, and thus $V'(w_s^n) - V'(w_s) = \gamma_s^1 + \gamma_s^2 \geq 0$, which since V is concave implies that it is optimal to set $w_s \geq w_s^n$. This implies also that $w_s > 0$ and therefore $\gamma_s^2 = 0$, and so $V'(w_s^n) - V'(w_s) = \gamma_s^1$. Moreover, when (ICEW) does not bind, $\gamma_s^1 = 0$ and therefore it is optimal to set $w_s = w_s^n$. If $w_s^n = 0$, then since $w_s \geq 0$, it follows again that $w_s \geq w_s^n$. If in addition, $w_s > 0$, then it must be that $\gamma_s^1 > 0$ and thus that (ICEW) binds. To see this, note that otherwise it would follow that $V'(w_s^n) - V'(w_s) = -\gamma_s^3 < 0$, and thus by the strict concavity of V that $w_s^n > w_s$, contradicting $w_s > w_s^n = 0$. Similarly, for part *(ii)*, from (42) it follows that $\frac{1-r}{r}\lambda_s^5 = \lambda_s^6$. Differentiating it and substituting $\lambda_s^{5\prime}$ and $\lambda_s^{6\prime}$ from (46), we obtain

$$h'(u_s) - h'(u_s^n) = \gamma_s^1 + \gamma_s^4 - \gamma_s^5 \tag{53}$$

[23] Since we are evaluating the *maximized* Hamiltonian, these conditions are satisfied by (e, u, u^n, v).

Since h is convex, by the same argument as above, we conclude that $u_s \geq u_s^n$, and that $u_s = u_s^n$ whenever (ICEW) does not bind.

For parts *(iii)* and *(v)*, differentiating $-\frac{1}{\beta}\lambda_s^2 + \lambda_s^5 = 0$ with respect to s, and then substituting for $\lambda_s^{2\prime}$ and $\lambda_s^{5\prime}$ from (44) and (46), we obtain

$$-V'(w_s) - \gamma_s^2 = h'(u_s) - \gamma_s^4 \tag{54}$$

If $u_s > 0$, then $\gamma_s^4 = 0$ and therefore $-V'(w_s) = \gamma_s^2 + h'(u_s) > 0$ implying that $w_s > 0$. It follows that $\gamma_s^2 = 0$, and so (54) becomes $-V'(w_s) = h'(u_s)$. If $u_s = 0$, then it must again be that $w_s > 0$; otherwise, by proposition 13*(i)*, it would follow that $w_s^n = 0$ and $u_s^n = 0$, which contradicts (ICE) since $e_s > 0$. Therefore, (54) becomes $-V'(w_s) - \gamma_s^2 = h'(0)$, implying $V'(w_s) > -h'(0)$, i.e., $w_s > \overline{w}$. For the converse in *(iii)*, if $w_s < \overline{w}$, i.e., if $V'(w_s) > -h'(0)$, then it must be that $u_s = 0$, since otherwise $-V'(w_s) = h'(u_s)$, which cannot be satisfied. Finally, for part *(iv)* of the proposition, from (41) and (42) it follows $-\frac{1}{\beta}\lambda_s^4 + \lambda_s^6 = 0$, which by differentiation and substitution implies

$$-V'(w_s^n) - \gamma_s^3 = h'(u_s^n) - \gamma_s^5 \tag{55}$$

The analysis is then similar to the one following (54) only that it is possible that $w_s^n = 0$.□

Proof of Proposition 14 If $r\left(\beta w_{\overline{s}} + u_{\overline{s}}\right) + (1-r)\left(\beta w_{\overline{s}}^n + u_{\overline{s}}^n\right) - c\left(\overline{s}, e_{\overline{s}}\right) - (1-r)\left(\beta w_s^n + u_s^n\right) > 0$ is satisfied for some value of s, by the continuity of w_s^n and u_s^n, that condition must hold on an interval $(s', s'') \subset [\underline{s}, \overline{s}]$, and and thus by the result of proposition 13 $w_s = w_s^n$ and $u_s = u_s^n$ for $s \in (s', s'')$. This implies $w_s' = w_s^{n\prime}$ and $u_s' = u_s^{n\prime}$ on (s', s''). By (ICTW) and $e_s' < 0$ it follows that $\beta w_s^{n\prime} + u_s^{n\prime} < 0$ and therefore that the condition $r\left(\beta w_{\overline{s}} + u_{\overline{s}}\right) + (1-r)\left(\beta w_{\overline{s}}^n + u_{\overline{s}}^n\right) - c\left(\overline{s}, e_{\overline{s}}\right) - (1-r)\left(\beta w_s^n + u_s^n\right) > 0$ is also satisfied for higher values of s. We conclude that once $w_s = w_s^n$ and $u_s = u_s^n$ for some value s, it must be that $w_{\widetilde{s}} = w_{\widetilde{s}}^n$ and $u_{\widetilde{s}} = u_{\widetilde{s}}^n$ for all $\widetilde{s} \in (s, \overline{s}]$. Denote by $\overleftrightarrow{s} \equiv \inf\limits_{s\in[\underline{s},\overline{s}]} \{s | w_s = w_s^n\}$. Then, (ICEW) does not bind for any $s \in (\overleftrightarrow{s}, \overline{s}]$, and therefore for these values, we have $w_s = w_s^n$, $u_s = u_s^n$ and $\beta w_s' + u_s' < 0$. For $s \in [\underline{s}, \overleftrightarrow{s})$, (ICEW) binds and thus $w_s > w_s^n$,

$u_s \geq u_s^n$ (with $u_s > u_s^n$ if $u_s > 0$) and $\beta w_s^n + u_s^n = \frac{1}{1-r}\left[r\left(\beta w_{\overline{s}} + u_{\overline{s}}\right) + (1-r)\left(\beta w_{\overline{s}}^n + u_{\overline{s}}^n\right) - c\left(\overline{s}, e_{\overline{s}}\right)\right]$, the latter implying $\beta w_s^{n\prime} + u_s^{n\prime} = 0$. (ICTW) implies then that $\beta w_s' + u_s' < 0$.

We will argue next that it must be that $w_s' < 0$ and $u_s' \leq 0$. As shown above, for all s, we have $\beta w_s' + u_s' < 0$. When $w_s \leq \overline{w}$, and thus $u_s = 0$, this immediately implies that $w_s' < 0$ and $u_s' = 0$. When $w_s > \overline{w}$, since $u_s > 0$, we have from the proof of proposition 13 that $V'(w_s) + h'(u_s) = 0$. This implies that w_s' and u_s' have the same sign. To see this, note that if, for instance, $w_s' > 0$ and $u_s' < 0$, then as $\widetilde{s}$ increases in a neighbourhood of s, both $V'(w_{\widetilde{s}})$ and $h'(u_{\widetilde{s}})$ strictly decrease, implying that $V'(w_{\widetilde{s}}) + h'(u_{\widetilde{s}}) = 0$ cannot hold. $\beta w_s' + u_s' < 0$ implies then that $w_s' < 0$ and $u_s' < 0$. Similarly, since as argued above, for all $s \in [\underline{s}, \overleftrightarrow{s})$, we have $\beta w_s^{n\prime} + u_s^{n\prime} = 0$, whereas from the same proof of proposition 13 we have that $V'(w_s^n) + h'(u_s^n) = 0$ when $w_s^n > \overline{w}$, it follows by a similar argument that $w_s^{n\prime} = 0$ and $u_s^{n\prime} = 0$. $\square$

Proof of Proposition 15 Differentiating $\lambda_s^1 + \lambda_s^2 \frac{1}{r\beta} c_e(s, e_s) = 0$ with respect to s, and then substituting for $\lambda_s^{1\prime}$ and $\lambda_s^{2\prime}$ from (43) and (44), we obtain

$$-y'(e_s) f(s) - V'(w_s) c_e(s, e_s) f(s) + \lambda_s^2 \frac{1}{r\beta} c_{es}(s, e_s) = 0 \tag{56}$$

Integrating $\lambda_s^{2\prime}$ from (44) between $\underline{s}$ and s while accounting for $\lambda_{\underline{s}}^2 = 0$ and for the fact that $\gamma_s^2 = 0$ for all s since $w_s > 0$, we obtain $\lambda_s^2 = -r\beta \int_{\underline{s}}^s \left[V'(w_\sigma) + \lambda^3\right] f(\sigma) d\sigma$. Applying the Dynamic Envelope Theorem to problem (34)-(40), we have that $V'(w) = -\lambda^3$. Substituting λ^3 into the expression obtained for λ_s^2 we obtain

$$\lambda_s^2 = r\beta \int_{\underline{s}}^s \left[V'(w) - V'(w_\sigma)\right] f(\sigma) d\sigma \tag{57}$$

Substituting λ_s^2 into (56), we conclude that (8) from the text of the proposition must hold. $\square$

Proof of Remark 16 From (57), we have $\lambda_s^{2\prime} = r\beta\left[V'(w) - V'(w_s)\right] f(s)$. Note that $\frac{d}{ds}\left[V'(w_s)\right] = V''(w_s) w_s' > 0$ since V is strictly concave and $w_s' < 0$ by proposition 14. Therefore $V'(w_s)$ is strictly increasing in s and thus there exists some $s' \in [\underline{s}, \overline{s}]$ such that $-V'(w) + V'(w_s) < 0$ if and only if $s > s'$. Since $f(s) > 0$, this implies $\lambda_s^{2\prime} < 0$ if and only if $s > s'$. Therefore, λ_s^2 is increasing for small values of s and then decreasing. Since from (47)-(48), we have $\lambda_{\underline{s}}^2 = 0$ and $\lambda_{\overline{s}}^2 = r\beta \int_{\underline{s}}^{\overline{s}} \gamma_s^1 f(s)ds \geq 0$, these imply that indeed $\lambda_s^2 > 0$ for all $s \in (\underline{s}, \overline{s})$, and therefore the remark follows from (57). $\square$

Proof of Proposition 18 The argument relies on the proof of the corresponding result for the static model in Barbos (2016). Thus, let $e_s = e_s^0$ and $w_s = w_s^n = u_s^n = 0$ for all $s \in [\underline{s}, \overline{s}]$, and let $u_s \equiv u_{\overline{s}} - \frac{1}{r} \int_s^{\overline{s}} c_e(\sigma, e_\sigma^0) e_\sigma^{0\prime} d\sigma$. By the same steps as in Barbos (2016), it follows that if $w = 0$, then

$$ru_{\overline{s}} = c\left(\overline{s}, e_{\overline{s}}^0\right) - \int_{\underline{s}}^{\overline{s}} c_s\left(s, e_s^0\right) F(s)ds \tag{58}$$

$u_{\overline{s}}$ needs to be non-negative because of the limited liability of the agent, which is precisely the assumption from the text of the proposition 18. As argued in Barbos (2016), $u_{\overline{s}} \geq 0$ is sufficient for all incentive constraints to be satisfied.[24] By the same argument as in the proof of proposition 10, since $\mathcal{A}$'s initial continuation value is 0, one can implement the first best effort in every period with the above contract by maintaining a continuation value equal to 0 at the beginning of every period. Since $\mathcal{A}$'s expected wage equals that under full information, it folllows then that the contract defined above is optimal since its value attains the theoretical upper bound. $\square$

Proof of Proposition 17 Using the multipliers from the definition of L, the per-period change in $\mathcal{P}$'s payoff from a small increase in r is

$$\begin{aligned}
\Delta V &= \int_{\underline{s}}^{\overline{s}} \frac{\partial}{\partial r} L\left(\cdot, s\right) f(s)ds \\
&= \int_{\underline{s}}^{\overline{s}} \left\{\left[\beta V(w_s) - h\left(u_s\right)\right] - \left[\beta V\left(w_s^n\right) - h\left(u_s^n\right)\right]\right\} f(s)ds + \int_{\underline{s}}^{\overline{s}} \lambda_s^2 \left[\frac{1}{r^2}\left(t_s + \frac{1}{\beta}k_s\right) - \frac{1}{\beta r^2} c_e(s, e_s) x_s\right] ds \\
&\quad + \int_{\underline{s}}^{\overline{s}} \left\{\lambda_s^3 \left[\left(\beta w_s + u_s\right) - \left(\beta w_s^n + u_s^n\right)\right] + \gamma_s^1 \left[\left(\beta w_{\overline{s}} + u_{\overline{s}}\right) - \left(\beta w_{\overline{s}}^n + u_{\overline{s}}^n\right) + \left(\beta w_s^n + u_s^n\right)\right]\right\} f(s)ds
\end{aligned}$$

[24] We employ here the underlying assumption that $e_s^0 > 0$ for all $s \in [\underline{s}, \overline{s}]$.

Using (36) and then (35), we have

$$\frac{1}{r}\left(t_s+\frac{1}{\beta}k_s\right)-\frac{1}{r\beta}c_e(s,e_s)x_s ds=-w_s'-\frac{1}{\beta}u_s'+w_s^{n\prime}+\frac{1}{\beta}u_s^{n\prime}$$

Using this and (44), integrating by parts, we have then

$$\int_{\underline{s}}^{\overline{s}}\lambda_s^2\left[\left(t_s+\frac{1}{\beta}k_s\right)-\frac{1}{\beta}c_e(s,e_s)x_s\right]ds=$$
$$=\left[\lambda_s^2\left(-w_s-\frac{1}{\beta}u_s+w_s^n+\frac{1}{\beta}u_s^n\right)\right]\Bigg|_{\underline{s}}^{\overline{s}}-\int_{\underline{s}}^{\overline{s}}\lambda_s^2\left(-w_s-\frac{1}{\beta}u_s+w_s^n+\frac{1}{\beta}u_s^n\right)f(s)ds$$
$$=r\beta\left(-w_{\overline{s}}-\frac{1}{\beta}u_{\overline{s}}+w_{\overline{s}}^n+\frac{1}{\beta}u_{\overline{s}}^n\right)\int_{\underline{s}}^{\overline{s}}\gamma_s^1 f(s)ds+r\beta\int_{\underline{s}}^{\overline{s}}\left[V'(w_s)+\lambda_s^3\right]\left(-w_s-\frac{1}{\beta}u_s+w_s^n+\frac{1}{\beta}u_s^n\right)f(s)ds$$

Substituting these results into the expression for ΔV derived above, and cancelling out terms, we have that

$$\begin{aligned}
\Delta V &= \int_{\underline{s}}^{\overline{s}}\left\{\left[\beta V(w_s)-h(u_s)\right]-\left[\beta V(w_s^n)-h(u_s^n)\right]\right\}f(s)ds+\\
&\quad+\int_{\underline{s}}^{\overline{s}}V'(w_s)\left(-\beta w_s-u_s+\beta w_s^n+u_s^n\right)f(s)ds+\int_{\underline{s}}^{\overline{s}}\gamma_s^1\left(\beta w_s^n+u_s^n\right)f(s)ds\\
&= \int_{\underline{s}}^{\overline{s}}\left[\beta V(w_s)-h(u_s)-\beta V(w_s^n)-h(u_s^n)-\beta V'(w_s)w_s\right]f(s)ds+\\
&\quad-\int_{\underline{s}}^{\overline{s}}V'(w_s)u_s f(s)ds+\beta\int_{\underline{s}}^{\overline{s}}\left[V'(w_s)+\gamma_s^1\right]w_s^n f(s)ds+\int_{\underline{s}}^{\overline{s}}\left[V'(w_s)+\gamma_s^1\right]u_s^n f(s)ds
\end{aligned}$$

Now, note first that from (54) and $\gamma_s^2=0$, we have $-V'(w_s)=h'(u_s)-\gamma_s^4$, so $-V'(w_s)u_s=h'(u_s)u_s$ since $\gamma_s^4u_s=0$ by the complementary slack condition. Second, from (52) we have $V'(w_s)+\gamma_s^1=V'(w_s^n)+\gamma_s^3$, so $\left[V'(w_s)+\gamma_s^1\right]\beta w_s^n=\beta V'(w_s^n)w_s^n$ since $\gamma_s^3w_s^n=0$. Finally, from (54) and then (53) we have $V'(w_s)=\gamma_s^4-h'(u_s)=\gamma_s^4-\left[\gamma_s^1+\gamma_s^4-\gamma_s^5+h'(u_s^n)\right]\Rightarrow V'(w_s)+\gamma_s^1=\gamma_s^5-h'(u_s^n)$. Therefore, $\left[V'(w_s)+\gamma_s^1\right]u_s^n=-h'(u_s^n)u_s^n$ since $\gamma_s^5u_s^n=0$. Collecting all these results, it follows that

$$\Delta V = \beta \int_{\underline{s}}^{\overline{s}} \left\{ \left[V(w_s) - V'(w_s) w_s \right] - \left[V\left(w_s^n\right) - V'\left(w_s^n\right) w_s^n \right] \right\} f(s) ds + \\ + \int_{\underline{s}}^{\overline{s}} \left\{ \left[h'(u_s) u_s - h(u_s) \right] - \left[h'\left(u_s^n\right) u_s^n - h\left(u_s^n\right) \right] \right\} f(s) ds \qquad (59)$$

Note now that the function $V(w) - V'(w)w$ is increasing in w since V is concave. Since from proposition 13, we have $w_s \geq w_s^n$, the first integral in (59) is thus positive. Similarly, the function $h'(u)u - h(u)$ is increasing in u since h is convex. Since $u_s \geq u_s^n$, the second integral in (59) is also positive. By the same argument as in the proof of proposition 9, since the per-period change to $\mathcal{P}$'s payoff from a small increase in r is positive, it follows that $V(w)$, which is the discounted sum of future period payoffs, is increasing in r. This completes the proof of proposition 17. □

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