Teaching the DiPasquale-Wheaton Model

Joseph S. DeSalvo

Department of Economics
College of Arts and Sciences
University of South Florida
4202 E. Fowler Ave. CMC342
Tampa, FL 33620-5700

Office: 813-974-6388
Fax: 813-974-6510
E-Mail: jdesalvo@usf.edu

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Abstract. The DiPasquale-Wheaton (1992) model graphically determines rental price, asset price, newly constructed stock, and total stock in a real estate market. Despite its frequent use in academic research, few textbooks exposit the model. We conjecture this is due in part to the difficulty of deriving its comparative static results. We derive a supply curve that simplifies graphical analysis and perform a complete graphical comparative static analysis. Although the main objective of this paper is to encourage pedagogical usage of the model, an appendix provides a mathematical derivation of the comparative static results, which has not heretofore appeared in the literature.

I. Introduction

Colwell (2002) reports that textbooks by DiPasquale and Wheaton (1996), Brueggeman and Fisher (1997), and Geltner and Miller (2001) contained the real estate model developed by DiPasquale and Wheaton (1992). The only recently published books we have found that do so are Pirounakis (2013) and Geltner, et al. (2014). Geltner, et al. (2014) provides the comparative static effects of a demand increase and a cap-rate decrease. Pirounakis only provides the four-quadrant graph to show the relationships among the model’s components. He performs no comparative statics. As Exhibit 1 documents, most textbooks neither cite nor use the model, a smaller number cite but make no use of the model, and, as already noted, very few both cite and use the model.
<table>
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<th>Year</th>
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Exhibit 1 (continued)

Textbook Usage of the DiPasquale-Wheaton Model

<table>
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<tr>
<th>Year</th>
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<td>2009</td>
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<td>Jacobus, 11th ed.</td>
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<td>Brueggeman &amp; Fisher, 14th ed.</td>
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<td>Floyd &amp; Allen, 10th ed.</td>
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<td>McKenzie, Betts &amp; Jensen, 6th ed.</td>
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<td>Goddard &amp; Marcum</td>
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<td></td>
<td>Geltner, Miller, Clayton &amp; Eichholtz, 3rd ed.</td>
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Despite the small number of textbooks, both in the past and present, that include discussion of the model, it has nevertheless been cited many times and featured in numerous academic articles. A Google Scholar search turned up 245 sources with citations to the DiPasquale-Wheaton article (1992) and 1,138 to the DiPasquale-Wheaton book (1996). Of the 245 article-citation sources, we eliminated those for which the following information was absent: title, English-language abstract, type of source (e.g., journal article, working paper, book, etc.), date, and citations in real estate textbooks, the latter of which are included in Exhibit 1. This left us with 175 citation sources. Exhibit 2 presents these citations by source and year. It is clear that the professional interest in the article continues and even grows.

Concentrating on the academic journal articles that cite the DiPasquale-Wheaton article, we find that most are citations with no or only a brief comment. A few are complimentary but make no real use of the model, e.g., Mills (1995) and Akimov, Stevenson, and Zagonov (2105). Some are complimentary and do use the model. Gat (2002, 5) says, “One of the best and most concise paradigms of the real estate market is the DiPasquale and Wheaton Four Quadrant model.” He also applies the model, as will be discussed below. Lisi (2015, 87) calls the DiPasquale-Wheaton model, “the most popular macroeconomic model of the housing market.” Some are pedagogical, usually providing an exposition of the model. For example, Achour-Fischer (1999) provides an interactive Excel version of the model, which employs specific functional forms and parameter values that allows the user to perform comparative static analyses.
<table>
<thead>
<tr>
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Source: Google Scholar (last accessed December 15, 2016)
*Includes conference papers.

The remaining articles employ the DiPasquale-Wheaton model in research. These articles fall into four broad categories: (1) those that use the model as a framework for explanation of historical real estate development; (2) those that use the model as a source of empirically testable hypotheses,
sometimes providing extensions; (3) those that provide extensions of the model, with only theoretical analysis; and (4) those that provide extensions and use the extended model in empirical applications.

The following articles are examples of category (1), those that use the model as a framework for explanation of historical real estate development. Renaud (1995) uses the model to study the Russian housing market during transition to a market economy. Hanink (1996) studies the geographical extent of office markets in the U.S. Eng and Lee (2003) investigate the impact of financing and securitization on real estate markets and urban development in Singapore. Allen, et al. (2004) study the transition of the Russian housing market from one in which the government produced most housing units to one in which the private market does so. Leung and Wang (2007) study the housing market in China, tracing the effect of various policies. Michelsen and Weiss (2010) study the development of the East German housing market after reunification. Lee, et al. (2014) examine the Korean housing market before and after macroeconomic fluctuations.

The following articles are examples of category (2), those that use the model as a source of empirically testable hypotheses, sometimes providing extensions. Kim and Yang (2006) estimate excess rates of return in the rental office market of Seoul. Constantinescu and Francke (2013), adding a lag structure to the model, track and analyze the development of the Swiss rental housing market. Wang and Kang (2014) study the effect on housing prices of China’s transition from a socialist system to a market economy. Lieser and Groh (2014) study how various socio-economic characteristics affect commercial real estate investment.
The following articles are example of category (3), those that provide extensions of the model, with only theoretical analysis. Colwell (2002) extends the model by endogenizing the cap rate; providing a non-proportional price-rent relationship; and introducing lagged adjustment, expectations, and vacancies. Colwell also introduces a long-run supply function, which is important to this paper and will be discussed later. Gat (2002) develops a dynamic version of the model, which includes speculative asset value, construction lag, and stock adjustment. Adding parameter values, he simulates a reduction in the stock of space, a sudden population increase, and a gradual increase in real income. These exercises are not applied to any real economy, but they do show how these effects work themselves out over time. After a thorough discussion of the model and its shortcomings, essentially following Colwell (2002), Lisi (2015) extends the model to include search and matching in the housing market.

The following articles are example of category (4), those that provide extensions and use the extended model in empirical applications. Viezer (1999) develops and estimates a real estate forecasting model for office markets, drawing on, among others, the DiPasquale-Wheaton model, and extending the model to include an endogenous cap rate and lags in adjustment. Gat (2000) provides specification, estimation, and simulation of the model and applies it to the Israeli housing market. Du Toit and Cloete (2004) develop a simulation model incorporating an equilibrium vacancy rate, and test it against data from the Pretoria, South Africa, and Perth, Australia, office markets. Chaney and Hoesli (2015) develop a dynamic version of the model and estimate it using Swiss multifamily residential data.
It is evident, therefore, that researchers find the model worthwhile for both theoretical and empirical analysis, while textbook authors generally do not choose to exposit it. We suspect that a reason for this is the difficulty of graphically determining the equilibrium values of the model’s endogenous variables under changes in the exogenous variables. This is a criticism noted by Colwell (2002).

It is also likely that the need for students to have some acquaintance with graphical supply and demand analysis causes textbook authors to eschew the model. All books discuss supply and demand, but relatively few use supply and demand curves, which we shall hereafter call simply supply-demand analysis.


Counting multiple editions only once, nine of these are real estate principles textbooks, for which the student audience might not be expected to have been exposed to supply-demand analysis, while five are real estate finance textbooks and two are real estate economics textbooks, for which it
might more likely be the case that students would have had some exposure to supply-demand analysis.

The DiPasquale-Wheaton model, as exposited here, uses one demand curve and two supply curves, in addition to the relation of rental to asset price through the cap rate and the relationship between the quantity of newly constructed stock and the existing stock through the depreciation rate. Therefore, expositing the DiPasquale-Wheaton model would not require much beyond what several textbooks already provide. Doing so, however, would expose students to a coherent real estate model widely used in theoretical and empirical analysis, a model that is missing from almost all real estate textbooks, even those that contain some supply-demand analysis.

The major purpose of this paper is to remove the difficulty of graphically obtaining the model’s comparative static results in the hope that this will encourage authors seriously to consider including a presentation of the DiPasquale-Wheaton model in future textbooks. Although that is our major objective, we also present in the appendix a complete mathematical derivation of the model’s comparative static results. This does not, to our knowledge, exist in the published literature. In addition to providing a solid theoretical basis for the graphical analysis, advanced real estate and urban economics courses and textbooks could use the mathematical exposition.

We attain our major objective by introducing a long-run supply curve into the graphical treatment of the model, a suggestion made by Colwell but apparently not taken up by others. This supply curve provides an “anchor” point for two of the endogenous variables from which the remaining endogenous variables may be determined. Although Colwell makes the suggestion, he does not
follow up with a rigorous analysis, only showing how to determine the four endogenous variables (essentially Exhibit 5 below) and providing a tabulation of comparative static results on the capital stock and rent.³

In addition to changes in demand for space (and stock), supply of new stock, and the cap rate, we also provide a comparative static analysis of the depreciation rate. This last result is ignored by DiPasquale and Wheaton (1992, 1996), probably because it was thought to generate the same qualitative results as the cap rate. It was included in Colwell’s tabulation of results, which conform to the results for rent and stock we derive. The results on new supply and asset price were not included in his tabulation, but we show they are in general ambiguous. We use an empirical argument to remove the ambiguity.

We proceed as follows: A review of the DiPasquale-Wheaton model, including derivation of a new long-run supply curve; a complete graphical comparative static analysis of the model; a brief summary; and a complete mathematical comparative static analysis in the appendix.

II. Review of the DiPasquale-Wheaton Model

A. The Relation among Demand, Existing Stock, Rent, and Price

Following DiPasquale and Wheaton (1992, 1996), we exposit a long-run model. Although changes in vacancies serve as signals that a long-run adjustment is required, we shall nevertheless ignore the vacancy component in our long-run analysis. It can be included, but it complicates the model without providing much insight, at least for an introductory pedagogical exposition.⁴
The left-hand panel of Exhibit 3, shows the capital-cost relation between \( p \), the price per unit of real estate space (here called the rental price or, simply, rent), and \( V \), the price per unit of real estate capital (here called the asset price or, simply, price), where \( \kappa \) is the “cap” rate, which capitalizes rent into price. The quantity of real estate space demanded, \( Q_d \), is a function of rent, \( p \). Demand and the currently existing capital stock, \( \bar{Q} \), determine the equilibrium rent, \( p^* \) and quantity \( Q^* \). The right-hand panel of Exhibit 3 shows this. The cap-cost relation converts rent into price, so the equilibrium price is \( V^* \), which is shown in the left-hand panel of Exhibit 3. We turn next to the supply of new stock and its relationship to existing stock.

Exhibit 3 Demand, Existing Stock, Rent, and Price

\[ p = \kappa V \]

\[ Q_d = D(p) \]

\[ Q^* = \bar{Q} \]

\[ V^* \]

\[ V \]

B. The Supply of New Stock and Its Relationship to Existing Stock

New real estate stock is supplied by the construction industry, and, in general, that industry’s supply function may be represented by \( Q_s = S(V) \), where the subscript \( s \) on \( Q \) indicates supply. The construction industry responds to the asset price of stock, not to its rental price, and, as usually assumed, the supply curve is upward sloping, implying an increasing-cost industry.\(^5\)
There exists previously built stock, which depreciates over time. If $\delta$ is the depreciation rate and $\bar{Q}$ is the existing stock, then $\delta \bar{Q}$ is the amount of stock that completely depreciates in any given period, e.g., a year. To maintain the stock, this amount must be replaced. If $Q_s$ is the quantity of new stock constructed in a year and if $\Delta \bar{Q}$ is the net addition to the stock, then $\Delta \bar{Q} = Q_s - \delta \bar{Q}$. In long-run equilibrium, there is no excess supply or demand, meaning that the existing stock provides all real estate space demanded. Hence, in long-run equilibrium, $\Delta \bar{Q} = 0$, so $Q_s = \delta \bar{Q}$, which requires that new construction replace fully depreciated stock.

Exhibit 4 illustrates these points. We draw the long-run new-stock supply curve in the left-hand panel of Exhibit 4, not in the usual way, but with the axes reversed. Its upward slope indicates that as the asset price increases, the quantity supplied increases, and vice versa. The right-hand panel shows the relation between the existing stock and the amount of new construction that is required to maintain that stock. We draw Exhibit 4 under the assumption that the values shown on the axes are long-run equilibrium values. This is something we must develop by putting Exhibit 3 and 4 together.
C. The Complete Model, Including Derivation of the Long-Run Supply Curve

Exhibit 5 provides a complete depiction of the model (ignore for now the upward sloping curve in the top-right quadrant). Given an initial stock, $Q^*$, and demand, $D$, the equilibrium rent is $\rho^*$. This rental price along with the given cap rate, $\kappa$, determines the equilibrium asset price $V^*$. The construction industry responds to this asset price by supplying $Q^*_s$, an amount that is just sufficient to replace the depreciated stock, $\delta Q^*$. Note that we are measuring the flow quantity demanded and supplied in the top-right quadrant as the same as the stock quantity, which means that one unit of stock produces one unit of space.

Although the model is complete, it is difficult to perform the comparative static analysis because one has to find by trial and error the equilibrating values of the variables. To make this easier to do, we develop another long-run supply curve, one that depends on the rental price and includes replacement as well as net additions to the stock ($Q_s = \delta \bar{Q} + \Delta \bar{Q}$). The new-stock supply curve in
the bottom-left quadrant of Exhibit 5 has \( Q_s \) and \( V \) on the axes, but we want one that has \( Q = \bar{Q} \) and \( p \) on the axes so that we can use it in the top-right quadrant. To obtain the desired supply curve, substitute \( p/\kappa \) for \( V \) and \( \delta \bar{Q} \) for \( Q_s \) in \( Q_s = S(V) \), getting \( \delta \bar{Q} = S(p/\kappa) \), or \( \bar{Q} = (1/\delta)S(p/\kappa) \), which is what we want. This is long-run supply, including both replacement of the fully depreciated stock.
and net additions to the stock. The equation of this supply function is redundant, in that the mathematical model does not need it. Its graph, however, is very important in that it eliminates the difficulty of obtaining the graphical comparative static results of the model, to which we now turn.

III. **Comparative Static Analysis of the Model**

A. **INTRODUCTION**

From a mathematical point of view, the model consists of the following four equations, to which we have added “shift parameters”:

\[ \bar{Q} = D(p, \theta_d) \]  
\[ Q_s = S(V, \theta_s) \]  
\[ p = \kappa V \]  
\[ Q_s = \delta \bar{Q} \]  

\( \theta_d \) represents a vector of demand determinants other than rent. \( \theta_s \) represents supply determinants other than asset price and expansion or contraction of the industry. Since we are assuming the new-stock supply curve is that of an increasing-cost industry, then we are embodying in the supply curve the increase in input prices as the industry expands (and the decrease as the industry contracts). We do not need to include shift parameters in the capital-cost equation or the stock-adjustment condition because their parameters are \( \kappa \) and \( \delta \).

Equations (1)–(4) are the structural equations of the model. The reduced form equations are the solution of the structural equations for the endogenous variables in terms of the exogenous variables:
Comparative static analysis consists in discovering how a change in one exogenous variable affects the endogenous variables, holding other exogenous variables constant. We shall do this graphically here, but the appendix provides the mathematics. In the graphical analysis, variables with subscript zeroes are the exogenous variables held constant, while those with subscripts 1 and 2 indicate the initial and final values of the endogenous variables and the one exogenous variable whose effects we are analyzing. Also, we shall only analyze the effect of an increase in an exogenous variable, but a decrease in that variable will reverse the qualitative effects of an increase.

B. AN INCREASE IN DEMAND

An increase in demand is a shift to the right of the demand curve in the upper right quadrant of Exhibit 6. Since the supply curve is unchanged, equilibrium rent and quantity increase from \( p_1 \) to \( p_2 \) and \( Q_{s1} \) to \( Q_{s2} \). The increase in the rental price increases its asset price from \( V_1 \) to \( V_2 \). The increased asset price induces firms to produce more stock along the new-stock supply curve, from \( Q_{s1} \) to \( Q_{s2} \). This increase in new stock must be sufficient to replace the fully depreciated stock and to provide a net increase in stock from \( Q_1 \) to \( Q_2 \).
C. AN INCREASE IN THE SUPPLY OF NEW STOCK

Exhibit 7 shows a supply increase in both the top-right quadrant and the bottom-left quadrant. In the top-right quadrant, we show the supply increase in the usual way as a shift of the supply curve to the right. In the bottom-left quadrant, however, the supply curve shifts to the left. This is due to the fact that we have reversed the axes from the usual presentation of supply.
Whatever the cause of the supply increase, the result is a decrease in equilibrium rent from $p_1$ to $p_2$, and an increase in equilibrium quantity, from $Q_1$ to $Q_2$, both of which are shown in the top-right quadrant. The decrease in rent results in a decrease in price from $V_1$ to $V_2$, shown in the top-left quadrant. The decrease in price would by itself cause a decrease in quantity supplied along the leftward-shifted supply curve in the bottom-left quadrant, but the increase in the supply curve
outweighs the downward movement along the curve, resulting in an increase in new construction from \( Q_{s1} \) to \( Q_{s2} \). It is easy to see that this must be the result from the change in \( Q \) in the top-right quadrant. The increase in new production replaces fully depreciated stock as well as resulting in a net increase in stock from \( Q_1 \) to \( Q_2 \).

**D. AN INCREASE IN THE CAP RATE**

Exhibit 8 shows the effects of an increase in the cap rate, due to a factor other than the depreciation rate. (Although the cap rate includes the depreciation rate as a component, a change in the depreciation rate is more complicated and we treat it separately later.) This affects both the supply curve in the top-right quadrant and the capital-cost line in the top-left quadrant. The capital-cost line pivots upward. An increase in the cap rate decreases \( p/\kappa = V \), other things equal, which shifts the supply curve in the upper right quadrant leftward, while being represented as a movement along the new-stock supply curve in the lower left quadrant.

Starting in the top-right quadrant, the increased cap rate decreases supply, which causes an increase in equilibrium rent from \( p_1 \) to \( p_2 \), and a decrease in equilibrium quantity from \( \bar{Q}_1 \) to \( \bar{Q}_2 \). By itself, the increase in rent would increase the asset price. In this case, however, the increase in the cap rate pivots the capital-cost line upward sufficiently to reduce the asset price. That this must be the case is seen in the bottom two quadrants, which show the decrease in quantity supplied required in the top-right quadrant. Production decreases because the asset price decreases. The net decrease
in production is brought about by not replacing fully depreciated stock and possibly by abandon- 
donment and demolition of not fully depreciated stock.

**Exhibit 8  Comparative Static Effects of an Increase in the Cap Rate**

\[
\begin{align*}
V_2 &\quad V_1 & V \\
p_2 &\quad p_1 & p \\
k_2V &\quad k_1V &\quad (1/\delta_0)S(p/k_2,\theta_{s0}) \\
k_1 &\quad (1/\delta_0)S(p/k_1,\theta_{s0}) &\quad D(p,\theta_{d0}) \\
Q_s &\quad S(V,\theta_{s0}) &\quad \delta_0\bar{Q} \\
Q_s_1 &\quad Q_s &\quad Q_s_2 \\
Q_s_1 &\quad Q_s &\quad Q_s_2 \
\end{align*}
\]
E. An Increase in the Depreciation Rate

Analysis of the deprecation rate is complicated by the fact that it appears in the equilibrium condition, $Q_s = \delta \bar{Q}$, and in the cap rate, $\kappa$, because the depreciation rate, $\delta$, is one of the terms in $\kappa$. It also appears in the long-run supply curve used in the upper-right quadrant of the graphical presentation of the comparative static results. Thus, when the deprecation rate changes it causes changes in three curves in Exhibit 9.

An increase in the rate of depreciation causes the long-run supply curve in the top-right quadrant to shift to the left, i.e., supply decreases. If $\delta_2 > \delta_1$, then $1/\delta_2 < 1/\delta_1$, which would shift the curve leftward. Note, however, that $\kappa_2 > \kappa_1$ because $\delta$ is one of the components of the cap rate. This also lowers $p/\kappa$, which would be a move down the supply curve in the bottom-left quadrant, but which would entail a leftward shift in the supply curve in the top-right quadrant.

The decrease in supply causes equilibrium rent to rise from $p_1$ to $p_2$ and equilibrium quantity to fall from $\bar{Q}_1$ to $\bar{Q}_2$. This implies a corresponding decrease in stock in the bottom-right quadrant of Exhibit 9 in conjunction with an upward pivot of the $\delta \bar{Q}$ curve due to the increase in the depreciation rate from $\delta_1$ to $\delta_2$.

So far, this is straightforward, but we still must explain the effects on $Q_s$ and $V$, which are unfortunately not straightforward. In general, the qualitative effects of an increase in the depreciation rate on these variables are ambiguous. In other words, it is possible to produce the effects on $p$ and $\bar{Q}$ with an increase in $V$ and a resulting increase in $Q_s$, which is what Exhibit 9 shows, or a decrease in $V$ and a resulting decrease in $Q_s$. To get the latter result, we must shift the supply curve in the
top-right quadrant less to the left than shown in Exhibit 9, so that $p$ still rises but produces a lower $V$ on the $\kappa_2 V$ curve in the top-left quadrant. This, in turn, will generate a lower $Q_s$ in the bottom-left quadrant.

**Exhibit 9  Comparative Static Effects of an Increase in the Depreciation Rate**

When we have a theoretically ambiguous comparative static result, to get an unambiguous result, we must either introduce a more restrictive assumption or use an empirical argument. We shall
take the latter approach. $V$ and $Q_s$ are directly related to $\delta$ if $\bar{Q} + \delta D_p V > 0$ (see the appendix), where $D_p$ is the rent-slope of the demand function, which is negative, while the other variables are positive. Under reasonable magnitudes for these variables, the magnitude of $\bar{Q}$ will greatly exceed that of the second term, so that the positivity of $\bar{Q}$ will outweigh the negativity of the other term.

To illustrate this, we adapt a numerical example from DiPasquale and Wheaton (1996, 8–10). The equations of the model in our notation are:

\begin{align*}
\bar{Q} &= \theta_d (400 - 10p) \tag{9} \\
Q_s &= 0.2(V - \theta_s) \tag{10} \\
p &= \kappa V \tag{11} \\
Q_s &= \delta \bar{Q}. \tag{12}
\end{align*}

Let $\theta_d = 10$ (million office workers per year, a demand determinant for office space), $\theta_s = 200$ (dollars per square foot of office space per year, a supply determinant of office space), $\kappa = 0.05$, and $\delta = 0.01$. Then, solving Equations (9)–(12) yields $\bar{Q} = 2.4$ billion sq. ft., $p = $16 per sq. ft. per year, $V = $320 per sq. ft., and $Q_s = 24$ million sq. ft. per year. Given these results and the assumed parameter values, $D_p = -100$, so $\bar{Q} + \delta D_p V = 2,400 + (0.01)(-100)(320) = 2,400 - 32 > 0$ (the unit in which both terms are measured is millions of sq. ft.).

Under this empirical assumption, we find that, in addition to $p$ being directly related to $\delta$ and $\bar{Q}$ being inversely related to $\delta$, we now have $V$ and $Q_s$ directly related to $\delta$. It may seem strange that, although the equilibrium stock contracts under an increase in the depreciation rate, the equilibrium amount of new construction increases. This means that to maintain the now lower quantity of stock,
builders must nevertheless build more new stock per year. Returning to our numerical example, suppose the depreciation rate rises from 0.01 to 0.02, then the new equilibrium $\bar{Q}$ is 1.75 billion sq. ft., and the new $Q_s$ is $(0.02)(1,750,000,000) = 35,000,000$ sq. ft. per year, which is larger than the 24,000,000 resulting from the lower depreciation rate.

**IV. Summary**

The main purpose of this paper is to kindle interest by textbook authors in the DiPasquale-Wheaton model as a pedagogical tool. We conjecture that one reason that only a couple of textbooks use the model is the difficulty of obtaining the comparative static results graphically. To address that problem, we adopt a suggestion of Colwell (2002) to include a supply curve in the upper right quadrant of the four-quadrant analysis. Such a curve captures both the construction of net new stock as well as replacement of completely depreciated stock. From a graphical viewpoint, it, coupled with the demand curve for space (and stock), provides an “anchor” point from which the two endogenous variables, rent and total stock, are determined. Given those two variables, it is a simple task graphically to determine the remaining two endogenous variables, asset price and newly constructed stock. We exposit a complete graphical comparative static analysis of the model.

The appendix presents a mathematical comparative static analysis of the model, not heretofore available. Among the comparative static results is one involving a change in the depreciation rate, which has not previously been treated graphically. We find that the effects of a change in the depreciation rate on the supply of new stock and its asset price are ambiguous in general. To remove the ambiguity, we posit an empirical argument.
Exhibit 10 summarizes the comparative static results of the model. A plus sign denotes a direct effect of the exogenous variable on the endogenous variable, while a minus sign denotes an inverse effect. Demand is the only case for which the exogenous effect is directly related to all of the endogenous variables. A shift in the new-stock supply curve is inversely related to rental and asset prices and directly related to new construction and the existing stock. A change in the cap rate (not due to depreciation change) is directly related to rent but inversely related to the other endogenous variables. A change in the depreciation rate is directly related to the rental and asset prices and to new construction but inversely related to the existing stock. The effects on $Q_s$ and $V$ depend on our assumption that the magnitude of $\bar{Q}$ is very large relative to $\delta D_p V$. Rental and asset prices move together in all cases except for a change in the cap rate.

<table>
<thead>
<tr>
<th>Exogenous Variable</th>
<th>Endogenous Variable</th>
<th>$p$</th>
<th>$Q_s$</th>
<th>$V$</th>
<th>$\bar{Q}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_d$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$\theta_s$</td>
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<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\delta$</td>
<td>+</td>
<td>$^{+a}$</td>
<td>$^{+a}$</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

$^{a}$For reasonable numerical values.
Appendix: Mathematical Comparative Static Analysis of the Real Estate Market Model

Preliminaries

The model consists of Equations (1)–(4) in the text. We begin by totally differentiating these equations, where a subscript on a function represents differentiation, getting:

\[ d\bar{Q} = D_p dp + D_{\theta_d} d\theta_d \]  
\[ dQ_s = S_V dV + S_{\theta_s} d\theta_s \]  
\[ dp = \kappa dV + Vd\kappa \]  
\[ dQ_s = \delta d\bar{Q} + \bar{Q}d\delta, \]

where \( D_p < 0, D_{\theta_d} > 0, S_V > 0, \) and \( S_{\theta_s} > 0. \) Rewriting these equations by isolating the exogenous variables and writing the resulting equation system in matrix notation, we obtain:

\[
\begin{bmatrix}
-D_p & 0 & 0 & 1 \\
0 & 1 & -S_V & 0 \\
1 & 0 & -\kappa & 0 \\
0 & 1 & 0 & -\delta
\end{bmatrix}
\begin{bmatrix}
 dp \\
dQ_s \\
dV \\
d\bar{Q}
\end{bmatrix}
=
\begin{bmatrix}
 D_{\theta_d} d\theta_d \\
S_{\theta_s} d\theta_s \\
Vd\kappa \\
\bar{Q}d\delta
\end{bmatrix}.  
\]  
\text{(A.5)}

Let \( \mathcal{D} \) be the determinant of the 4 × 4 matrix above. Upon evaluating that determinant, we have

\[ \mathcal{D} = S_V - \delta D_p \kappa > 0. \]  
\text{(A.6)}

Comparative Statics of \( p \)

\[
\mathcal{D} dp = \begin{vmatrix}
D_{\theta_d} d\theta_d & 0 & 0 & 1 \\
S_{\theta_s} d\theta_s & 1 & -S_V & 0 \\
Vd\kappa & 0 & -\kappa & 0 \\
\bar{Q}d\delta & 1 & 0 & -\delta
\end{vmatrix} = \kappa \bar{Q} d\delta + S_V Vd\kappa - \kappa S_{\theta_s} d\theta_s + \delta \kappa D_{\theta_d} d\theta_d.  
\]  
\text{(A.7)}
\[
\frac{\partial p}{\partial \theta_d} = \frac{\delta \kappa D \theta_d}{D} > 0, \quad \frac{\partial p}{\partial \theta_s} = -\frac{\kappa S \theta_s}{D} < 0, \quad \frac{\partial p}{\partial \kappa} = \frac{S \nu V}{D} > 0, \quad \frac{\partial p}{\partial \delta} = \frac{\kappa \bar{Q} + S \nu V}{D} > 0. \quad (A.8)
\]

In this and the following results for a change in \( \delta \), the second term in the numerator appears because \( d \kappa = d \delta \).

**Comparative Statics of \( Q_s \)**

\[
\mathcal{D} dQ_s = \begin{vmatrix}
-D_p & D_{\theta_d} d \theta_d & 0 & 1 \\
0 & S_{\theta_s} d \theta_s & -S \nu & 0 \\
1 & V d \kappa & -\kappa & 0 \\
0 & \bar{Q} d \delta & 0 & -\delta
\end{vmatrix} = S \nu \bar{Q} d \delta - \delta \kappa D_p S \theta_s d \theta_s + \delta S \nu D \theta_d d \theta_d + \delta D_p S \nu V d \kappa.
\]

\[
(A.9)
\]

Hence,

\[
\frac{\partial Q_s}{\partial \theta_d} = \frac{\delta S \nu D \theta_d}{D} > 0, \quad \frac{\partial Q_s}{\partial \theta_s} = -\frac{\delta \kappa D_p S \theta_s}{D} > 0, \quad \frac{\partial Q_s}{\partial \kappa} = \frac{\delta D_p S \nu V}{D} < 0, \quad \frac{\partial Q_s}{\partial \delta} = \frac{S \nu \bar{Q} + \delta D_p S \nu V}{D} < 0.
\]

\[
(A.10)
\]

**Comparative Statics of \( V \)**

\[
\mathcal{D} dV = \begin{vmatrix}
-D_p & 0 & D_{\theta_d} d \theta_d & 1 \\
0 & 1 & S_{\theta_s} d \theta_s & 0 \\
1 & 0 & V d \kappa & 0 \\
0 & 1 & \bar{Q} d \delta & -\delta
\end{vmatrix} = -S_{\theta_s} d \theta_s + \bar{Q} d \delta + \delta D_p V d \kappa + \delta D_{\theta_d} d \theta_d. \quad (A.11)
\]

Hence,

\[
\frac{\partial V}{\partial \theta_d} = \frac{\delta D \theta_d}{D} > 0, \quad \frac{\partial V}{\partial \theta_s} = -\frac{S \theta_s}{D} < 0, \quad \frac{\partial V}{\partial \kappa} = \frac{\delta D_p V}{D} < 0, \quad \frac{\partial V}{\partial \delta} = \frac{\bar{Q} + \delta D_p V}{D} < 0. \quad (A.12)
\]
**Comparative Statics of \( \tilde{Q} \)**

\[
\mathcal{D}dV = \begin{vmatrix} -D_p & 0 & 0 & D_{\theta_d}d\theta_d \\ 0 & 1 & -S_V & S_{\theta_s}d\theta_s \\ 1 & 0 & -\kappa & Vd\kappa \\ 0 & 1 & 0 & \tilde{Q}d\delta \end{vmatrix} = \kappa D_p \tilde{Q}d\delta + D_p S_V Vd\kappa + S_V D_{\theta_d}d\theta_d - \kappa D_p S_{\theta_s}d\theta_s. \quad (A.13)
\]

Hence,

\[
\frac{\partial \tilde{Q}}{\partial \theta_d} = \frac{S_V D_{\theta_d}}{\mathcal{D}} > 0, \quad \frac{\partial \tilde{Q}}{\partial \theta_s} = -\frac{\kappa D_p S_{\theta_s}}{\mathcal{D}} > 0, \quad \frac{\partial \tilde{Q}}{\partial \kappa} = \frac{D_p S_V V}{\mathcal{D}} < 0, \quad \frac{\partial \tilde{Q}}{\partial \delta} = \frac{\kappa D_p \tilde{Q} + D_p S_V V}{\mathcal{D}} < 0. \quad (A.14)
\]

**Signs of \( \partial Q_s/d\delta \) and \( \partial V/d\delta \)**

Both of these terms are positive if \( \tilde{Q} + \delta D_p V > 0 \). We argued in the text that \( \tilde{Q} \) is very likely to be much larger in magnitude than \( \delta D_p V \), in which case \( \tilde{Q} + \delta D_p V > 0 \).
Endnotes

1. Fisher (1992) independently developed a similar model, but he did not provide graphical or mathematical comparative static analysis. Colwell (2002, 24) notes that the Brueggeman-Fisher text used Fisher’s version of the model.

2. It is included in Lisi (2015) but only as a review of Colwell (2002).

3. Colwell (2002) provides several other “tweaks” to the model, namely, distinguishing between the capitalization rate and the reciprocal of the gross income multiplier; endogenizing the capitalization rate; and introducing short-run adjustments, expectations, and vacancies. Lisi (2015) reviews these extensions, and he goes on to integrate the model with search and matching models.


6. It is not clear what geographical unit DiPasquale and Wheaton have in mind. Ten million office workers is a large number, but if we scale it down to one thousand office workers, nothing changes except that the two terms are now measured in thousands of square feet. A $\bar{Q}$ of 2.4 million square feet of office space is about half the size necessary to meet one criterion of Garreau’s definition of an edge city (1991, 6).
References


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