Interdependent Voting in Two-Candidate Voting Games

Abstract

The election of a political candidate is a public good for all those who prefer it and a public bad for those who are opposed. Given free-rider problems and other features of collective action, the probability that any one voter will vote for a preferred candidate is unlikely to be independent of the voting probabilities of all others with the same electoral preference, as is so often assumed. Interdependent voting probabilities are, therefore, likely to be the norm rather than the exception. We make no assumption about this interdependence other than a generalized concavity condition.

The point of departure for this paper is independent voting in the context of a two-candidate, regular concave voting game. Such games always have a unique, asymptotically stable equilibrium platform. This platform is not the ideal point of the median voter, however. It is also, in general, not Pareto optimal. With interdependent voting, the unique equilibrium is preserved when candidates have the same initial endowments. If one candidate is advantaged (so the game is not symmetrical), however, plurality-maximizing candidates will cycle endlessly.

A candidate advantage creates a convex set of platform choices, none of which can be defeated by the disadvantaged opponent. An equilibrium without convergence is achieved if we assume that the advantaged candidate chooses from an undefeatable set a platform that maximizes utility while an opponent maximizes her plurality. The set of undefeatable platforms collapses on a unique winning platform as the advantage disappears.

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Interdependent Voting in Two-Candidate Voting Games

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How do voters interact in support of the election of candidates they prefer? This is a neglected topic in public choice economics and the oversight turns out to be of considerable importance.[1] Interdependent voting can wreak havoc in voting games that otherwise have unique and stable equilibriums. With even the simplest interaction, candidate plurality functions can have multiple peaks and plurality-maximizing politicians can get caught up in endless cycles.

In the modern era, interest in voting theory was sparked when Arrow (1963) drew attention to the paradox of voting, a condition under which no alternative can win a majority vote over all others.[2] For a while, it was unclear whether the paradox was a curiosity or something to be routinely expected. With single-peaked preferences in one dimension, voters' ideal points can be arrayed on a line along which the median voter's preference dominates all alternatives (Hotelling 1929; Downs 1957; Black 1958; Enelow and Hinich 1984). However, about two decades ago it became clear that in multiple dimensions instability is the rule, not the exception,

[1] There have been a number of models of campaign contributions, including Ben-Zion and Eytan (1974), Welch (1984) Bental and Ben-Zion (1975) and Welch (1980). None of these, however, investigated the impact of campaign contributions on the existence and stability of voting equilibrium. Also see Cook, et. al.(1974), and Sankoff and Mellos (1972).

[2] The phenomenon had been discovered almost two hundred years earlier by Condorcet (1785) and one hundred years later by Dodgson (1876). For a discussion of these and other early contributions see Black (1958), Riker (1961), and Mueller (2003)

and that voting cycles will typically range all over the issue space (Ordeshook and Shepsle, 1982).[3]

These conclusions, however, were based on one-person-one-vote models in which voting outcomes are dependent on preferences alone. Because there is nothing voters can do to increase or decrease their level of support for a favored candidate, these models lacked the concept of marginalism, which so imbues the rest of economic theory.[4] One way to incorporate the marginalist principle is to allow voter abstention and to assign to each voter a probability of voting which varies, depending on the voter's interest in the outcome. Hinich, Ledyard and Ordeshook, (1972) showed that this modification is sufficient to generate a unique equilibrium at which competing candidates converge to the same platform under reasonable assumptions.[5]

The introduction of marginalism into voting theory was short-lived, however. Over the past 25 years or so a new literature has emerged on probabilistic voting.[6] Today, probabilistic voting usually means that candidates do not know exactly how voters will vote. The probabilities, therefore, are properties of the state of knowledge of candidates, rather than characteristics of voters themselves (Coughlin, 1991). Under assumptions similar to those we incorporate herein, conventional probabilistic voting models have unique, stable equilibriums. But they pay two heavy prices for this result. First, the models break down for small voting

[3] A particularly depressing judgment was rendered by Riker, who concluded that politics, not economics, "is *the* dismal science because we have learned from it that there are no fundamental equilibria to predict." (1982, p. 19).

[4] The principle of marginalism in this context holds that participants in the political system can always incrementally increase or decrease the level of effort they are willing to make to affect the political outcome, and these incremental changes always have an impact on political equilibrium, no matter how small (Goodman, 1976).

[5] They justified their assumptions about voter behavior based on the alienation and indifference hypotheses (see Hinich, Ledyard and Ordeshook 1972, and Davis, Hinich and Ordeshook 1970). For other formulations of this approach, see Hinich, Ledyard and Ordeshook (1973), McKlvey (1975), Hinich (1977) and Ledyard (1984).

[6] See, e.g., Enelow and Hinich, 1981, 1984; Ordeshook, 1971; Coughlin, 1982, 1986, 1991; Coughlin and Nitzan, 1981a, 1981b; Coughlin, Mueller, and Murrell, 1990; Feldman and Lee, 1988; and the surveys in Mueller, 2003 at p. 263; and in Lafay, 1993.

groups where the uncertainty assumption collapses. Second, they relegate the voter to a completely passive role. The voter's only function is to have preferences (Coughlin, 1990).

The single, most important feature of all democracies is that elections are collective consumption goods. Every candidate's election is a public good for all those who favor it and a public bad for those who are opposed. As a result, people have an incentive not to vote, relying on others of similar interest to make the effort to elect a preferred candidate (Downs, 1957). Among the group of supporters, this form of free-riding can be overcome to some degree by peer pressure in the form of persuasion, personal appeals, participation in get-out-the-vote efforts, and other overt actions to influence electoral outcomes (Olson, 1965). To the extent that people act in this way, and are successful, any one voter's behavior will not be independent of the behavior of all others.

Accordingly, we introduce the concept of a vote production function to link the efforts of citizens to the vote totals they produce for the candidates they prefer. Note: our interest here is not in *how* voters interact with each other, but *that* they interact. We make no assumption about the interaction other than a generalized concavity condition.[7]

In Section I we model electoral competition as a two-person concave voting game. In Section II we consider the special case of a linear vote production function (independent voting), and show that every regular concave voting game has a unique, asymptotically stable equilibrium at which both candidates converge to the same platform. We show that the same equilibrium

[7] The observation that votes depend on more than voter preferences - for example, candidate popularity or perceived trustworthiness (Ansolabehere and Snyder, 2000; Groseclose, 2001; Schofield, 2003 & 2004), campaign advertising (Ashworth, 2006; Prat, 2002, Ansolabehere and Iyengar, 1996; Gerber, 1998) and party affiliation (Adams, 1998) – implies that votes are generated by a process that combines the inputs of platform promises, money, political association, and valence among other things. For our purposes in this paper we assume the various inputs can be summarized by a single numeraire input we call supporter or voter "effort." The law of diminishing marginal returns motivates the assumption that the production of votes is concave over effort.

persists even if there is asymmetry that gives a natural advantage to one of the two candidates. Further, if the candidates are constrained to take different positions on some issues, they will converge in all issue dimensions where they are not constrained.

In Section III we evaluate the welfare consequences of the equilibrium points for regular concave voting games. Against the heartening discovery that the political process has a determinate outcome is the disheartening discovery that the outcome will almost never be Pareto optimal. This conclusion is similar to the results we found for the regulation of output and price, the production of public goods and government supply of other goods and services (Goodman and Porter 1985, 1988 and 2004).

In Section IV, we show that in the face of nonlinear vote production functions (interdependent voting) candidate plurality functions can have multiple peaks and vote cycling can occur. Nonetheless, such games can have a unique equilibrium at which the two candidates may converge. When asymmetry is introduced, however, the voting game will not have an equilibrium for plurality-maximizing candidates, but will instead lead to endless cycles. This finding strongly suggests that instability in democratic voting is more likely to arise from the way in which voter preferences are translated into votes than from the distribution of the preferences themselves.[8]

These results cause us to question three conventional assumptions about voting games: (1) that candidates maximize pluralities, (2) that a Nash equilibrium is the appropriate equilibrium concept and (3) that candidates engage in conventional platform adjustments when they are not in equilibrium. In Section V, we set aside all three assumptions and demonstrate

[8] Non-convergence in stochastic models of voting can be introduced by voter uncertainty (Berger, Munger and Pothoff, 2000). Convergence away from the electoral center is predicted in models with an advantaged candidate (Ansolabehere and Snyder, 2000; Groseclose, 2001; Schofield, 2003).

that an advantaged candidate has access to a set of points that cannot be defeated by an opponent.[9] Moreover, this set is a convex set that converges to the unique winning platform described in Section IV as the advantage disappears.

In Section VI we consider what a candidate might do with an advantage. Maximizing utility over the set of undefeatable platforms always produces a unique platform choice. However, if the disadvantaged candidate maximizes her plurality, the candidates' platforms will differ.

In the long run, candidate advantages are likely to disappear, making the competition symmetrical.[10] In this case, democratic voting is likely to produce a unique winning platform that is independent of the preferences of the candidates themselves. Significantly, this unique winning platform is not necessarily the ideal point for the median voter. It is instead a platform that equates the "marginal political prices" groups of voters on either side of each issue are willing to pay to obtain a dollar's worth of benefit from the political system. This result is consistent with our own previous work as well as the work of Becker (1983) and Peltzman (1984).

I. The Voting Game

We model the political system as a two-person, zero sum, symmetric, non-cooperative game. Candidate 1 and candidate 2, denoted by superscripts 1 or 2, adopt the platforms $X^1 = (x_1^1, x_2^1, \dots, x_n^1)$ and $X^2 = (x_1^2, x_2^2, \dots, x_n^2)$, respectively. Each platform issue, x_i, is chosen from issue space χ, an n-dimensional compact, convex set. We assume that for each voter, *j*,

[9] Feld and Gorfman (1991) demonstrate this for advantaged candidates who enjoy voter loyalty.
[10] In two party competitions one party might possess an advantage in any given election that derives from a candidate's seniority, wealth, or name recognition but in the long run these advantages disappear.

there is a benefit function $\mathrm{B}^j(x_1, x_2, \ldots, x_n)$ which is a positive, concave function, with ∇_{xx}, the Hessian of second-order derivatives of B^j, negative definite.[11] Our benefit functions are fully analogous to voter utility functions, commonly used in voting models. The difference is one of interpretation. The standard approach is to assume that voters derive utility from political platforms. By contrast, we assume that voters derive utility from states of the world created by those platforms. This allows us to make the kind of welfare judgments familiar to economists in other contexts.[12]

We denote by $L^j(X^1, X^2)$ the support voter *j* will give to a candidate with platform X^1, given an opponent who endorses platform X^2. Similarly, $L^j(X^2, X^1)$ is the support voter *j* will give to a candidate with platform X^2, given an opponent who endorses X^1. We assume throughout that *L* is a uniform measure of support, regardless of the preferred candidate, and that voters always support the candidate whose platform they prefer. Thus:

$$\text{(2a)} \qquad L^j(X^1, X^2) = \begin{cases} l^j[\mathrm{B}^j(X^1), \mathrm{B}^j(X^2)] & \text{if } \mathrm{B}^j(X^1) > \mathrm{B}^j(X^2) \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{(2b)} \qquad L^j(X^2, X^1) = \begin{cases} l^j[\mathrm{B}^j(X^2), \mathrm{B}^j(X^1)] & \text{if } \mathrm{B}^j(X^2) > \mathrm{B}^j(X^1) \\ 0 & \text{otherwise.} \end{cases}$$

We assume that the function L^j is continuous and the function l^j is continuous and twice differentiable with

[11] For the proof of the following theorems, it is not necessary for every benefit function to be concave. However, they must be strictly concave on the average.

[12] Imagine the platform issues are regulated product prices. We do not assume that voters get utility from the prices. Instead, the prices give rise to consumer and producer surplus. Further, the optimal vector of prices is the one that maximizes the total surplus (See Goodman and Porter 1988).

(3) $l_1^j > 0,\ l_2^j < 0,\ l_{11}^j \le 0,\ l_{22}^j \ge 0.$

(4) $\lim\limits_{B_1 \to B_{2+}} l_1^j = -\lim\limits_{B_2 \to B_{1-}} l_2^j$

We model the support that voters give to candidates rather than the votes they give for several reasons. First, in a world of campaign finance, the potential influence a voter can have on an election extends well beyond the mere act of voting. Second, unlike probabilistic votes, support is not constrained to the closed interval [0, 1] and, therefore, not subject to the impossibility result identified by Kirchgassner (2000). Finally, the fact that votes are produced by combining the support of individuals, rather than simply summing over individual preferences, is the insight that motivates the findings of this paper.

If voters act independently of each other, the sum of support is analogous to the sum of the voting probabilities and our model yields the same results as the probabilistic voting model.[13] However, when voters interact their efforts are inputs into a process that creates votes for the candidate they prefer. Further, the electoral value of a voter's effort on behalf of one candidate (given the candidate's coalition of supporters) may not equal the electoral value of that same effort when dedicated to another candidate (given the second candidate's coalition of supporters).

We denote by the function *V(L)* the transformation of support by voters into actual votes for the candidates whose platforms they prefer and offer the following:

Definition: A vote production function identifies a candidate's unique vote total, given the cumulative support of all voters who prefer the candidate's election to that of an opponent.

[13] Note that the measures of support L^j can be mapped onto the interval [0, 1] and preserve the cardinal properties of support by simply dividing each individual's support by the maximum individual support offered.

Although we call this function a vote production function, it differs from its counterpart in standard production theory. Unlike an entrepreneur or a firm, candidates do not control all of the ways in which inputs combine to produce an output. In fact, in the model used here candidates have only one function: to select platforms. By contrast, voters themselves "produce" votes through their actions and interactions with each other.

The way in which preferences are translated into votes is undoubtedly complex. The election of a candidate is a public good for all the supporters and a public bad for all the opponents and, therefore, suffers from expected free-rider problems. Also, the composition of the groups of voters who favor or oppose a candidate will tend to change whenever a platform changes. It is not our goal here to explore these complexities. Rather in what follows we show that anything more complicated than a linear relationship is likely to produce endless cycling.

Each candidate is assumed to maximize an expected plurality (the difference between the candidate's own votes and the votes for the opponent) given by:

(5a) $$\Phi^1\left(X^1, X^2\right) = V\left(\sum_j L^j(X^1, X^2)\right) - V\left(\sum_j L^j(X^2, X^1)\right)$$

(5b) $$\Phi^2\left(X^2, X^1\right) = V\left(\sum_j L^j(X^2, X^1)\right) - V\left(\sum_j L^j(X^1, X^2)\right)$$

The game will be said to have a Nash equilibrium if neither candidate can improve her position by any unilateral move. That is, (X^{1*}, X^{2*}) is an equilibrium if:

(6a) $$\Phi^1(X^{1*}, X^{2*}) \geq \Phi^1(X^1, X^{2*}) \quad \forall\, X^1$$

(6b) $$\Phi^2(X^{2*}, X^{1*}) \geq \Phi^2(X^2, X^{1*}) \quad \forall\, X^2$$

At a point other than equilibrium we assume the two candidates continuously adjust their platforms at constant rates, such that:

(7a) $dx_i^1 / dt = r^1 \partial \Phi^1 / \partial x_i^1 \qquad i = 1, \ldots, n \qquad r^1 > 0$

(7b) $dx_i^2 / dt = r^2 \partial \Phi^2 / \partial x_i^2 \qquad i = 1, \ldots, n \qquad r^2 > 0$

We consider other adjustment assumptions below.

II. Independent Voting

For the case of a linear vote production function, (with independent voter behavior), we have:

(8) $V(L^i) = L^i,$ where $L^1 = \sum_j L^j (X^1, X^2)$ and $L^2 = \sum_j L^j (X^2, X^1).$

In other words, a candidate's vote total is given by summing the support of all who favor his or her election.

Definition: Any game described by conditions (1) through (8) is a regular concave voting game.

Following the lead of Hinich, Ledyard and Ordeshook (1972), we offer two theorems and three corollaries.

Theorem I: For every regular concave voting game, (a) there is a unique equilibrium (X^{1*}, X^{2*}) *and (b) from any initial point in the issue space, say* $\left(X_0^1, X_0^2\right)$*, a solution to equation set (7),* $\left(X^1(t), X^2(t)\right)$*, exists and converges to* $\left(X^{1*}, X^{2*}\right)$*, i.e.* $\lim_{t \to \infty} X^1(t) = X^{1*}$ *and* $\lim_{t \to \infty} X^2(t) = X^{2*}$.

Proof: Goodman's theorem (1980) states that every game possesses a unique and asymptotically stable equilibrium provided that each player's payoff (plurality) function is strictly concave in his own strategy (Hessian is negative definite) and convex in the strategies of all the opponents (sum of the Hessians is positive semi-definite) and the sum of all payoff functions is concave.[14] From the fact that ∇_{xx} is negative definite, it follows that $\nabla_{X^1X^1}$ is negative definite and $\nabla_{X^2X^2}$ is positive definite for candidate 1, and $\nabla_{X^2X^2}$ is negative definite and $\nabla_{X^1X^1}$ is positive definite for candidate 2. Since the sum of the payoff functions is always zero, the conditions of Goodman's theorem are met. Q.E.D.

Theorem II: The equilibrium (X^{1*}, X^{2*}) is a point at which the two candidates converge in every issue dimension, i.e., $x_i^{1*} = x_i^{2*} = x_i^*$ for all i.

Proof: Note that from the requirement of symmetry, if (X^{1*}, X^{2*}) is an equilibrium, (X^{2*}, X^{1*}), (X^{1*}, X^{1*}) and (X^{2*}, X^{2*}) must also be equilibriums. But since the equilibrium is unique, we must have $X^{1*} = X^{2*}$. In equilibrium, both candidates endorse the same platform and $X^{1*} = X^{2*} = X^*$.

How important is symmetry? One way to violate the symmetry condition is to give one candidate an initial endowment of resources (core support), L_0, that is not available to the other candidate. Another way is to assume that because of biased voting, one candidate is the recipient

[14] Goodman's Theorem is a special case of a set of theorems proved by Rosen (1965). The assumptions of concave/convex payoff functions are reasonable for a broad class of voting games. But they are not defensible for every game. Coughlin (1986) and Feldman and Lee (1988) have shown that the required restrictions are unreasonable in the context of unconstrained redistribution of income and Kirchgassner (2000) has shown that they may break down in other cases. Ball (1999) has shown that they are also unreasonable if candidates have goals other than vote maximization or plurality maximization.

of more support than an opponent, even if the two platforms are identical. A third way is to place constraints on the platforms of the two candidates.

Corollary I: If one candidate has initial resources, L_0*, not available to the opponent, the equilibrium point* (X^*, X^*) *will be unaffected.*

Corollary II: If $L^1 \neq L^2$ *when* $X^1 = X^2$*, the equilibrium point* $\left(X^*, X^*\right)$ *will be unaffected.*

Corollary III: If the two candidates are constrained on issue p so that $x_p^1 \neq x_p^2$ *then in equilibrium* $x_i^1 = x_i^2 = x_i^*$ *for all* $i \neq p$*.*

Theorem II says there is one platform that can defeat all others in a majority vote. The three corollaries say that even with asymmetry two candidates will always converge at the winning platform unless they are explicitly constrained from doing so.

Proof: Note that in introducing non-symmetry we have not altered any of the conditions required for the proof of Theorem I. So we can continue to be assured of a unique equilibrium. Consider any platforms, *A* and *B*, defined over the m-dimensional space $(m \leq n)$ that delineates the choices of candidates on issues where they are not constrained. Let $\Phi^1(A, A) = a$ and $\Phi^2(A, A) = -a$ when the candidates do converge and assume there is an equilibrium *(B, A)*, $B \neq A$, where they do not. Then we must have $\Phi^1(B, A) = -\Phi^2(A, B) = a$. Otherwise one of the two candidates could improve his or her plurality by simply matching the opponent's position. Hence $\Phi^1(B, A) = \Phi^1(A, A) = a$. But from the strict concavity of Φ^1 we must have

$\Phi^1\left(\delta B+\left(1-\delta\right)A, A\right)>\Phi^1\left(B, A\right)$, for $0<\delta<1$, contradicting the assumption that the point *(B, A)* is an equilibrium.[15] Q.E.D.

In what follows, we assume symmetry unless otherwise indicated.

III. Welfare Consequences

For an interior solution, the first order conditions for a maximum for each candidate require:

$$\text{(9)} \qquad \sum_j l_1^j \frac{\partial B^j}{\partial x_i^k} - \sum_j l_2^j \frac{\partial B^j}{\partial x_i^k} = 0 \qquad\qquad i=1,\ldots,n \text{ and } k=1,2$$

at the equilibrium point X^*. Note that l_1^j is the marginal support voter *j* is willing to give a favored candidate for an extra dollar's worth of benefit from the political system. Thus condition (9) says that the cumulative support per dollar of expected benefit for those who favor an increase in x_i must be equal to the support per dollar of expected benefit for all those opposed. At the equilibrium platform, "political prices" offered by opposing groups of voters must be equal.

Note that the equilibrium platform does not necessarily represent the preferences of the median voter and in the general case will almost never do so. In fact, unlike the median voter result in one-man-one-vote deterministic models, the equilibrium platform described here is influenced by the preferences of every voter and his willingness to act. Also, any change in the

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[15] Φ^1 is strictly concave in any subset of the issues. To see this, array the issues so that those in the subset come first in $\nabla_{X_1X_1}$. Because the principle minors of $\nabla_{X_1X_1}$ alternate in sign beginning with $\left|M_1\right|<0$, the sufficient condition for strict concavity in the subset of issues is met.

preferences or the marginal support of any voter will change the equilibrium, reflecting the marginalist principle.

What are the expected welfare consequences of the equilibrium points in concave voting games?

Theorem III: The platform (X^{1*}, X^{2*}) *is Pareto optimal if* $l_1^j = -l_2^j = \lambda$ *for all voters.*

Proof: Note that if $l_1^j = -l_2^j = \lambda^j$ for every voter, then condition (9) reduces to the first order condition for maximizing $\sum_j \mathrm{B}^j(x_1, x_2, \ldots, x_n)$, the sum of the benefit functions for all voters. A similar argument applies to constrained solutions. Q.E.D.

According to Theorem III, optimality is ensured if the marginal effort is the same for all voters. But since there is no mechanism that induces such equality, we are left with the pessimistic conclusion that optimality will almost never occur. Plurality maximizing candidates will always make Pareto adjustments if they can compensate the losers for their losses. But such compensation will generally require individualized tax rates and/or benefit payments. And since no extant political system has such individualized programs, optimality is not an expected outcome, at least in large-scale majority voting systems.[16]

IV. Interdependent Voting

We now consider a nonlinear function $V(L)$ with interdependent voter behavior, where

[16] It is common practice to regard majority voting equilibrium as Pareto optimal, since if it were possible to make one voter better off without making some other voter worse off one of the candidates would surely do so. As shown in Goodman and Porter (1985), this observation is true but trivial. Global optimality occurs only if the choice set open to politicians is the same as the choice set theoretically available to society as a whole. But this assumption is unwarranted. In conventional economics, non-optimal results occur because economic actors find that they are unable to compensate the losers for their losses. The same thing happens in politics. On other examples of equating electoral equilibrium with a social welfare optimum, see Coughlin (1982), Coughlin and Nitzan (1981a), Lindbeck and Weibull (1987), and Mueller (2003 at p. 253).

(10) $V_L > 0$

(11) $V_{LL} < 0.$[17]

In general, nonlinerity means the plurality functions will not be differentiable at the points where $x_i^1 = x_i^2$. The reason: when the candidates have not converged on issues other than i, they will tend to have different quantities of the input *L*. Thus, the marginal product of a contribution will be different, depending on whether the point $x_i^1 = x_i^2$ is approached from the left or the right. Additionally, the plurality functions will not be everywhere concave. They may have multiple peaks and valleys and they may not be stable. Nonetheless, a unique equilibrium will still exist, provided that the candidates have the same initial endowment.

Theorem IV: With nonlinear vote production described by equations (10) and (11), and $L_0^1 = L_0^2$, the platform X^* is a unique equilibrium for the game.

Proof: Theorem 1 implies that $L^1(X^*, X) > L^2(X, X^*)$ for all $X \neq X^*$. Since *V* is monotonic in *L*, $V(L^1) > V(L^2)$ whenever $L^1 > L^2$. It follows that the platform X^* can defeat any other platform in a majority vote. Hence, (X^*, X^*) must be an equilibrium for the game. Q.E.D.

Non-linearity by itself affects neither the winning platform nor the welfare properties of that platform. The plurality function for candidate 1 is:

(12) $$\Phi^1 = V\left(L_0 + \sum_j L^j((X^1, X^2)) \right) - V\left(L_0 + \sum_j L^j(X^2, X^1) \right).$$

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[17] One way an individual can give more support than a single vote is by providing information and applying peer pressure that increases the vote support of others. As in conventional production theory we assume diminishing marginal returns to support. If there are increasing marginal returns, the following conclusions still hold but the role of leader and follower when there is a dominant candidate will be reversed.

When $X^1 = X^2 = X^*$ both plurality functions are continuously differentiable. Hence, the derivatives of Φ^1 at (X^*, X^*) exist and the first-order necessary conditions for the maximization of (12) that define an interior solution at (X^*, X^*) are:

$$(13) \qquad \frac{dV}{dL}(\sum_j l_1^j \frac{\partial \mathrm{B}^j}{\partial x_i}) - \frac{dV}{dL}(\sum_j l_2^j \frac{\partial \mathrm{B}^j}{\partial x_i}) = 0 \quad i = 1, \ldots, n.$$

Since L_0 is the same for both candidates, equation (13) is equivalent to equation (9).

Let us again introduce asymmetry by positing a difference in initial endowments. Then we have:

Theorem V: For V(L) nonlinear, $L_0^1 \neq L_0^2$, *there exists no equilibrium for the game.*

Proof: Without loss of generality assume candidate 1 has the larger initial endowment, $L_0^1 > L_0^2$, and consider points where the candidates have the same platform. For a marginal increase in platform plank x_i candidate 2 will receive a differential contribution dL^+ and her opponent will receive dL^-. From condition (4) we know that if candidate 2 instead chooses a marginal decrease in x_i she will receive dL^- and her opponent will receive dL^+. Either $dL^+ \geq dL^-$ or $dL^+ \leq dL^-$. If the former, $d\Phi^2 = V_L(L_0^2)dL^+ - V_L(L_0^1)dL^- > 0$. Since candidate 2, with a smaller endowment, has a higher marginal product at a point of convergence [i.e., $L_0^1 > L_0^2$ implies that $V_L(L_0^1) < V_L(L_0^2)$], candidate 2 can increase her plurality by choosing a higher value of x_i. If the latter case, $d\Phi^2 = V_L(L_0^2)dL^- - V_L(L_0^1)dL^+ > 0$ and candidate 2 can gain from a marginal decrease in x_i. Hence, no point of convergence can be an equilibrium.

Let c be candidate 1's plurality at any point of convergence, and consider a possible equilibrium at $X^1 \neq X^2$ where the candidates do not converge. Then we must have $\Phi^1\left(X^1, X^2\right) = -\Phi^2\left(X^2, X^1\right) = c.$ Otherwise, one of the two candidates could increase her plurality by simply matching the platform of the opponent. Thus the move $\left(X^1, X^2\right) \Rightarrow \left(X^1, X^1\right)$ maintains candidate 2's plurality and from $\left(X^1, X^1\right)$ we know she can increase her plurality by a subsequent move. Hence, no divergent point $\left(X^1, X^2\right)$ can be an equilibrium for the game. Q.E.D.[18]

An example of candidate 2's incentives is depicted in Figure I, where there is a single issue and voters are grouped around two ideal points. Beginning at a point of convergence Candidate 2 can gain by a movement in either direction. Note that the adjustment equations (7) do not guarantee that she will move in the direction that maximizes her plurality, however. Assume that both candidates always choose a plurality-maximizing platform, given a choice by the opponent. The resulting cycling is depicted in Figure II, where the cycles function like an oscillating super nova. Beginning at x^* we experience wide swings which expand, contract and expand again.[19]

Three properties of this example are interesting, especially if they hold for more complex cases. First, with concave vote productions a challenger always improves her plurality by moving away from the dominant candidate. Second, the dominant candidate improves his

[18] Aragones and Palfrey (2002, 2004) demonstrate this for two candidate competition over a single issue. The advantaged candidate has the incentive to copy the disadvantaged candidate while the latter has the incentive to move away from the former.
[19] Figures I and II are the result of a stylized example. The voter benefits, voter contributions, endowments, and the vote production function together with a narrative are detailed in the appendix.

plurality by chasing the challenger. Third, the pursuit of a Nash equilibrium requires substantial flip flopping on the issues, as the cycles range over wide portions of the issue space.

V. Electoral Dominance

Some voters may penalize a candidate who flip-flops, just on principle. So, we are interested in alternatives to myopic plurality maximization. If one candidate must pick a platform and stick with it, regardless of the actions of an opponent, what platform should she pick? Since in asymmetric voting games, one candidate will tend to have a natural advantage over an opponent, we introduce the following:

Definition: Candidate 1 is a dominant candidate if there exists any platform X, $X \neq X^*$, *such that* $\Phi^1\left(X, X^2\right) \geq 0, \forall X^2$.

Importantly, dominance expands the range of options for the favored candidate since a dominant candidate can adopt platforms other than X^* and not be defeated. Let

(14) $$\Gamma(X) = \left\{X : V(X, X^2) - V(X^2, X) \geq 0, \forall X^2\right\}$$

be the set of platforms for dominant candidate 1 that cannot be defeated. Then

Theorem VI. (a) $\Gamma(X)$ *is a closed convex set defined by the endowments,* L_0^1 *and* L_0^2*, (b)* $\Gamma(X : L_0^1, L_0^2) \subset \Gamma(X : L_1^1, L_0^2)$ *if* $L_0^1 < L_1^1$*, and (c)* $\Gamma(X) \to X^*$ *as* $L_0^1 \to L_0^2$.

Proof: (a) Assume $\Gamma(X)$ is not convex. This implies that there exists at least two platforms for candidate 1, say $\hat{X}$ and $\tilde{X}$, in $\Gamma(X)$ and some X_0^2 and some $0 < \delta < 1$ such that $\Phi^1\left(\tilde{X}, X_0^2\right) > 0$ and $\Phi^1\left(\tilde{X}, X_0^2\right) > 0,$ but $\Phi\left(X', X_0^2\right) < 0$ where $X' = \delta\hat{X} + (1-\delta)\tilde{X}$. Now consider the advantage of candidate 1, given X_0^2. From Theorem I, the differential support

$\sum_j [L^j(X, X_0^2) - L^j(X_0^2, X)]$ is a concave function of *X*. Hence, the greater-than-or-equal-to set

$H(X : X_0^2, L_0^1, L_0^2) \equiv \left\{X : L^1(X, X_0^2) - L^2(X_0^2, X) \geq L_0^2 - L_0^1\right\}$ is a convex set, containing all platform choices for candidate 1, given X_0^2, for which her support plus her endowment are greater than or equal to candidate 2's support plus his endowment. Points in *H* are candidate 1 platforms that cannot be defeated by X_0^2. Since $\hat{X}$ and $\tilde{X}$ must be in *H*, by its convexity so must X', contradicting the assumption that X_0^2 can defeat X'. Including equal vote production, closes the set.

(b) $H(X : X_0^2, L_0^1, L_0^2) \subset H(X : X_0^2, L_1^1, L_0^2)$ for every value of X_0^2 when $L_0^1 < L_1^1$. Therefore, $\Gamma(X, L_0^1, L_0^2) \subset \Gamma(X, L_1^1, L_0^2)$.

(c) From (b) we know that $\Gamma(X)$ shrinks as $L_0^1 \to L_0^2$ and by Theorem IV we know that when the endowments are equal the only platform that cannot be defeated is X^*. Q.E.D.

The boundary of $\Gamma(X)$ is $\left\{X : L^1(X, X^{2*}) - L^2(X^{2*}, X) = L_0^2 - L_0^1\right\}$, where X^{2*} is the plurality-maximizing platform for candidate 2 given the platform of candidate 1. Boundary points can result in electoral ties. Candidates who wish to assure election will choose to be on the interior of $\Gamma(X)$. We therefore define the set

$\Gamma(X : \alpha) = \left\{X : V(X, X^2) - V(X^2, X) \geq \alpha, \forall X^2\right\}$. By the logic employed in the proof of *Theorem VI*, $\Gamma(X : \alpha)$ is also a closed, convex set. Positive values of α yield a set of points that assure the election of the dominant candidate by a plurality greater than or equal to α.[20] When there is uncertainty, being near the boundary of $\Gamma(X)$ is risky since candidates may misperceive

[20] The set $\Gamma(X : \alpha)$ shrinks as α increases. For high values of α, $\Gamma(X : \alpha)$ is empty.

the scope of their advantage. Additionally, there are often unforeseen surprises in elections that can reduce a candidate's advantage. Uncertainty and an aversion to risk will make the dominant candidate retreat to the interior of $\Gamma(X)$. A dominant candidate does not maximize his plurality by choosing X^*. But given any uncertainty about L_0^1 and L_0^2 the platform X^* assures his success. Thus, the platform X^* continues to be of interest, even with dominant candidates.

VI. Candidate Utility Functions

Again without loss of generality, let candidate 1 be the dominant candidate and let

(14) $U^1(X) = U^1(x_1, x_2, ..., x_n) \; \forall x \in \Gamma(X : \alpha)$

be candidate 1's utility function, defined over the set of platform choices at which she maintains a minimum plurality and cannot be defeated. Like the voter benefit functions, we assume this function is strictly concave.

Theorem VII: If a dominant candidate maximizes utility (equation 14) over the undefeatable set $\Gamma(X : \alpha)$ and if *the dominated candidate maximizes a plurality (equation 5b), the game will have an equilibrium* (X^{1*}, X^{2*}) *at which* $X^{1*} \neq X^{2*}$ *and* X^{1*} *is unique.*

Proof: Note that X^{1*} is independent of X^2 and since $U^1(x)$ is a strictly concave function defined over a convex set, X^{1*} must be unique. For a given X^1, $\phi^2(X)$ is a continuous function defined over a convex set and so must reach a maximum somewhere, although the maximum may not be unique. That $X^{1*} \neq X^{2*}$ follows from the proof of Theorem V. Q.E.D.

Theorem VII is interesting because in most elections there probably is a dominate candidate, who is free to choose among a set of undefeatable platforms. Further, any serious challenger to such a candidate is likely to maximize his plurality (since not doing so would enlarge the margin of his expected defeat). This result has several important consequences. First, the result may explain why we never get complete convergence in any election, even though we also do not observe widespread cycling. Second, Theorem VII frames the unresolved debate over capture versus ideology in politics. (Kau and Rubin, 1979; Kalt and Zupan, 1984, 1990; Peltzman, 1984, 1985; Davis and Porter, 1989) Theorem VII states that when there is a dominant candidate she cannot choose a platform outside $\Gamma(X)$ without inviting defeat (capture) but within $\Gamma(X)$ she is free to pursue her ideology. Finally, when there is uncertainty in an electoral competition one can think of α as determined by the dominant candidate's aversion to risk. More risk averse candidates will want greater assurance that they will prevail in the election and will choose a larger α (expected margin of victory). This increased security comes at the expense of platform preferences and may be thought of as a risk premium the candidate must pay.

In the long run, democratic voting is likely to be symmetrical because, given enough time, what one candidate can do, any other candidate can do. In the long run, convergence at the unique equilibrium platform is the norm. In the short run, however, dominant candidates and asymmetry are to be expected. Dominant candidates can afford to be ideologues and platform differences are expected

Conclusion

The point of departure for this paper is the regular concave voting game. With independent voting behavior, such games have a unique winning platform, X^*, that can defeat all other platforms in a majority vote. In general, this platform is not the ideal point of the median voter; nor is it likely to be Pareto optimal. Even if the game is nonsymmetrical, X^* continues to be the winning platform, although candidates may not find it with conventional adjustment mechanisms.

With interdependent voting, the results are less benign. Since voter interaction to support a candidate of choice is likely to be complex, we expect vote production functions to be nonlinear. With both asymmetry and nonlinearity, moreover, there will be no Nash equilibrium and plurality-maximizing candidates will cycle endlessly. Provided that the interaction among voters is well-behaved, however, the dominant candidate will have access to a set of platforms, including X^*, that cannot be defeated by an opponent. Within this set, a dominant candidate may choose a platform that reflects ideological preferences. If the opponent is a plurality maximizer, however, the two candidates will not converge.

Risk averse candidates will choose platforms closer to X^* and as dominance diminishes, both candidates will converge to X^*.

Appendix

Consider two groups of voters, *A* and B, with benefit functions

$$B^A = x^{1/2}$$
$$B^B = (1000 - x)^{1/2}$$

defined over issue space $X = [0, 1000]$. Candidate 1 offers platform $x^1 \in X$ and candidate *2* offers platform $x^2 \in X$. Groups contribute the difference in benefits to the preferred candidate according to

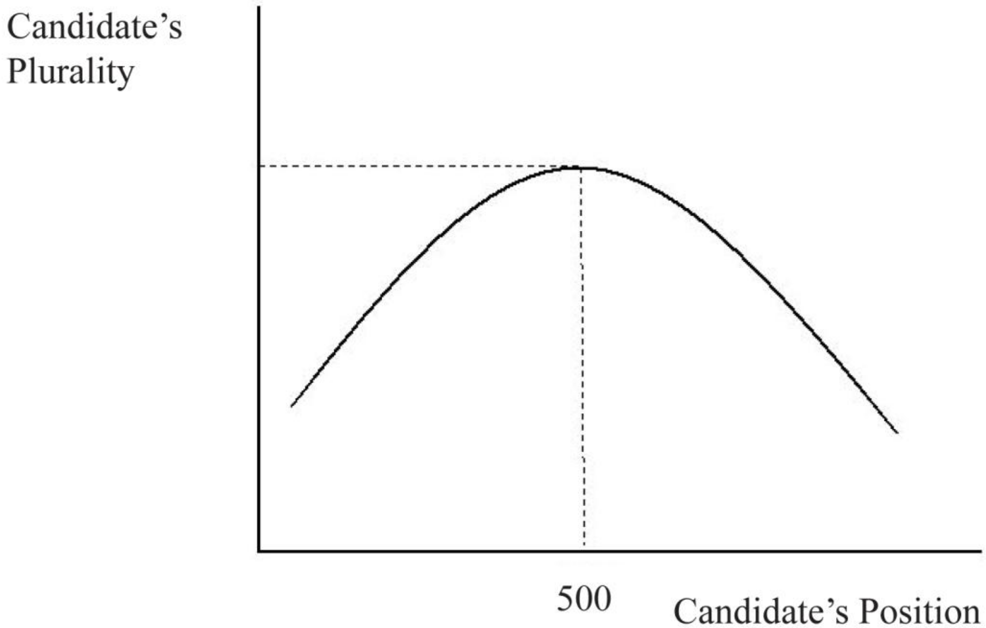
$$L^j = \begin{cases} B^i(x^j) - B^i(x^{\sim j}) & \text{if } B^i(x^j) > B^i(x^{\sim j}) \\ 0 & \text{otherwise} \end{cases} \quad i = A, B \quad j = 1, 2.$$

Let votes be produced from the endowment of candidates, L_0^j, and the contributions of supporters, L^j, according to $V^j = (L_0^j + L^j)^{1/2}$, so that the plurality functions are

$$\Phi^1 = -\Phi^2 = (L_0^1 + L^1)^{1/2} - (L_0^2 + L^2)^{1/2}.$$

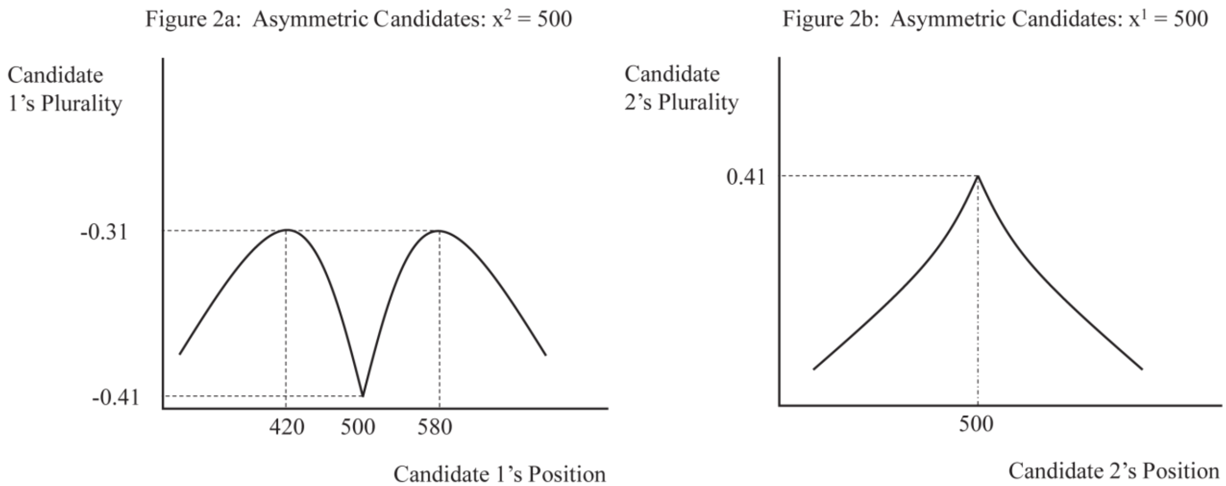
Consider the case of symmetry with original endowments $L_0^1 = L_0^2 = 1$. The equilibrium point is $x^{1*} = x^{2*} = 500$. The plurality function of either candidate, opposed by a candidate who adopts the equilibrium platform, is reproduced graphically in Figure 1. This is a stable equilibrium, since neither candidate can increase her plurality by a unilateral move. Notice too that the plurality functions are smooth, with continuous first derivatives.

Figure 1: Symmetric Candidates' Plurality



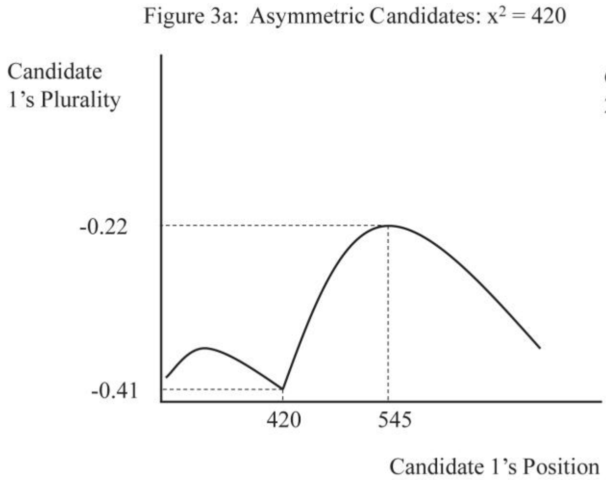
Now consider the case where candidate 2 has an endowment advantage, with $L_0^1 = 1$ and $L_0^2 = 2$. Figures 2a and 2b show the plurality functions of Candidates 1 and 2, respectively, given an opposing platform, $x = 500$. When both candidates adopt this platform, candidate 2 wins with a plurality of 0.41 (= $2^{1/2}$ – 1). But the platform is no longer stable because for candidate 1, the vote-producing input has a higher marginal product. The ideal platform for candidate 1 is now 420 or 580, with plurality -0.31.

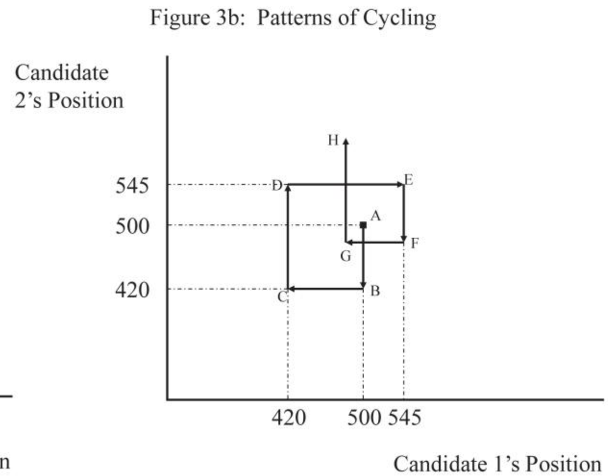
Unstable Platforms With Asymmetric Endowments



Suppose candidate 1 moves to platform $x^1 = 420$. Candidate 2, still at $x^2 = 500$, would have a plurality of 0.31 and could improve to 0.41 by matching candidate 1 at $x^2 = 420$. Candidate 1's plurality function when candidate 2 offers platform 420 is presented in Figure 3a. Candidate 1 now maximizes her plurality by moving to $x^1 = 545$. Of course, candidate 2 can reestablish his lead by matching candidate 1 at this new platform and the cycle will continue. The pattern of cycling is presented in Figure 3b. Several points are illustrated by this example. First, the plurality functions when candidates are not symmetrically endowed are no longer smooth at the switching point where the candidate platforms are the same because the contributions from supporters have different values for the two candidates. Second, the lesser-endowed candidate will always seek to differentiate herself from the better-endowed candidate and the latter will be inclined to follow. Third, the better-endowed candidate cannot be defeated if he remains at $x^2 = 500$ and permits a higher plurality for an optimizing opponent if he strays from this platform. Finally, platforms cycle around 500, exploding outward when both candidates are close to this platform and dampening back toward this platform when they are farther away.

Cycling With Asymmetric Endowments





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